square plate when the inner electrode has different positions represented by

$$\lambda = 20x_0/a = 20y_0/a = 1, 2, 3, 4, \text{ and } 5.$$

λ	1	2	3	4	5
$2\pi\sigma t R$	4·7808 3·6803	4·7284 3·6267	4·6412 3·5380	4·5177 3·4129	4·3539 3·2472

For any value of b/a the resistance decreases as the inner electrode is moved from the centre towards an edge. It decreases also in the case of the square plate when the inner electrode is moved from the centre towards a corner. The effect on the resistance is relatively greater, however, when the radius of the inner electrode is increased; it is true, for all values of b/a and for all the electrode positions considered here, that the resistance is reduced by about one-quarter when the small radius is trebled. On the other hand, a movement of the electrode from the centre of the plate to a point half-way between the centre and one of the longer edges (a movement of 30 diameters when $\delta = 1/450$) causes a fall in the resistance of less than one-twelfth.

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NOTE ON A CLASS OF SOLUTIONS OF THE NAVIER–STOKES EQUATIONS REPRESENTING STEADY ROTATIONALLY-SYMMETRIC FLOW

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[Received 4 July 1950]

SUMMARY

This note describes one- and two-parameter families of solutions of steady rotationally-symmetric viscous flow. The solutions are such that the Navier–Stokes equations reduce to ordinary differential equations in a single position variable. The one-parameter family represents flow which is rigid-body rotation at infinity and over a plane through the origin; the solution given by von Kármán in 1921 is one member of this family. The two-parameter family represents flow which is rigid-body rotation over each of two planes at a finite distance apart. The case of large Reynolds number is particularly interesting, since the two bounding planes are then separated by a region of rigid-body rotation and translation in which viscous effects are negligible.

Introduction

T. v. Kármán (1) has pointed out a simple solution of the Navier–Stokes equations of motion which describes the steady flow of a viscous fluid in a semi-infinite region bounded by an infinite rotating disk. This 'solution' is not yet given analytically, since one is left with two ordinary non-linear differential equations in a single independent variable which must be solved numerically, but to have carried a solution of the Navier–Stokes equations even so far by exact analysis was (and still is) something of a novelty. Moreover, the solution has the very interesting property that it is also a solution of the appropriate boundary-layer equations, the terms neglected in boundary-layer theory being identically zero for this type of motion.

The purpose of this note is to show that there are one- and two-parameter families of solutions having the particular mathematical simplicity of Kármán's solution; Kármán's solution is one member of the one-parameter family. In Kármán's problem the flow far from the disk is assumed to be wholly normal to the disk and to be induced by the rotation of the disk. In general, if other conditions far from the disk are assumed, the particular simplicity of Kármán's solution is lost, but it will be shown below that the simple form of the solution is retained if the fluid at infinity has an arbitrary uniform angular velocity γ about the axis of rotation of the disk. If ω is the angular velocity of the disk, there is found to be a solution for each value of γ/ω between $-\infty$ and $+\infty$. Kármán's solution corresponds to

$\gamma/\omega = 0$, whereas $\gamma/\omega = \mp\infty$ describes a flow which is rotating uniformly at infinity and which is bounded by a stationary disk. In this latter case there is thus the opportunity of describing quantitatively the tendency for particles of sugar to migrate to the centre of a cup of tea which has been stirred. The qualitative explanation (**2**) of this phenomenon is well known, of course; the new point is that the flow away from the disk induced by the rotation is *uniform* over planes parallel to the disk, just as the flow towards the disk is uniform over these planes in Kármán's problem.

A two-parameter family of solutions of the same simple type describes the flow between two parallel infinite disks which are rotating about the same axis with different angular velocities; in addition to the ratio of the angular velocities of the disks, the Reynolds number based on the distance bet een the disks is now a relevant parameter. A numerical method of determining the flow field in the special case in which one disk is stationary and the Reynolds number of rotation of the other disk is small has recently been described by Casal (**3**).

A family of solutions of related type has been described by Miss Hannah (**4**). This family is obtained by combining the flow towards the disk produced by a source at infinity on the axis of rotation with the rotating flow induced by the disk. Kármán's solution is obtained when the source-flow is made zero, and, at the other limit, if the disk is stationary the viscous stagnation-point solution described by Homann (**5**) is recovered. This family of solutions will not be included in the following discussion.

The governing equations

If v_r, v_θ, v_z are velocity components in the directions of increase of cylindrical polar coordinates r, θ, z, and p is the pressure, the Navier–Stokes equations of steady motion of a fluid of density ρ which is symmetrical about the axis $r = 0$ are as follows:

$$v_r\frac{\partial v_r}{\partial r}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\nabla^2 v_r-\frac{v_r}{r^2}\right), \tag{1}$$

$$v_r\frac{\partial v_\theta}{\partial r}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_\theta}{r} = \nu\left(\nabla^2 v_\theta-\frac{v_\theta}{r^2}\right), \tag{2}$$

$$v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 v_z, \tag{3}$$

where
$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}.$$

The continuity equation is

$$\frac{1}{r}\frac{\partial r v_r}{\partial r}+\frac{\partial v_z}{\partial z} = 0. \tag{4}$$

The plane $z = 0$ is identified with the plane of a uniformly rotating disk (angular velocity ω) so that one set of boundary conditions is

$$v_z = v_r = 0, \qquad v_\theta = \omega r, \quad \text{at} \quad z = 0. \tag{5}$$

The other set of boundary conditions will depend on the problem under discussion; if the fluid is unbounded in the z-direction the conditions to be assumed are

$$v_r \to 0, \quad v_\theta \to \gamma_1 r, \quad \text{as} \quad z \to \infty, \tag{6}$$

whereas if there is a second rotating disk at $z = d$ the conditions are

$$v_z = v_r = 0, \qquad v_\theta = \gamma_2 r, \quad \text{at} \quad z = d, \tag{7}$$

where γ_1, γ_2, and d are disposable constants. This excludes Miss Hannah's family of solutions, which give a radial velocity at $z = \infty$, but it includes the one- and the two-parameter families mentioned in the introduction.

The simple property of Kármán's solution and of the solutions sought herein is that the flow normal to the disk is uniform over planes parallel to the disk. That is,

$$v_z \equiv v_z(z),$$

and as a consequence, from (4), and assuming that v_r is finite at $r = 0$,

$$v_r = -\frac{r}{2}\frac{dv_z}{dz}. \tag{8}$$

Equation (3) can then be integrated to give

$$p/\rho = \nu\frac{dv_z}{dz}-\tfrac{1}{2}v_z^2+\Pi(r). \tag{9}$$

From equation (1) we find that the arbitrary function $\Pi(r)$ satisfies

$$\frac{1}{r}\frac{d\Pi(r)}{dr}-\frac{v_\theta^2}{r^2} = \text{function of } z \text{ only}.$$

The boundary condition (5) then requires

$$\Pi(r) = \tfrac{1}{2}r^2(\omega^2+c), \tag{10}$$

where c is a constant, so that

$$v_\theta/r = \text{function of } z \text{ only}. \tag{11}$$

Equations (1) and (2) then become

$$\left(\frac{1}{2}\frac{dv_z}{dz}\right)^2-\tfrac{1}{2}v_z\frac{d^2v_z}{dz^2}-\left(\frac{v_\theta}{r}\right)^2 = -(\omega^2+c)-\frac{\nu}{2}\frac{d^3v_z}{dz^3} \tag{12}$$

and
$$-\frac{dv_z}{dz}\frac{v_\theta}{r}+v_z\frac{d(v_\theta/r)}{dz} = \nu\frac{d^2(v_\theta/r)}{dz^2}. \tag{13}$$

In the case of the semi-infinite fluid with boundary conditions (5) and (6), the asymptotic form (as $z \to \infty$) of (12) is

$$c = \gamma_1^2-\omega^2 \tag{14}$$

and only five boundary conditions are needed for the solution of (12) and (13). These are supplied by (5) and (6), the uniform axial velocity V at $z = \infty$ being determined as part of the solution. In the case of the two rotating disks all six of the boundary conditions (5) and (7) are needed to solve (12) and (13) and to determine the constant c.

The form of solution which has been assumed is still valid when there is a uniform suction through the surface of either or both of the disks, the boundary condition $v_z = 0$ being then replaced by $v_z =$ a prescribed constant. Suction through the surface of the disks will give rise to some interesting modifications of the flow patterns described below, but will not be considered further.

The variables may be made non-dimensional by using $(\nu\omega)^{\frac12}$ as a reference velocity and $(\nu/\omega)^{\frac12}$ as a reference length, the direction of positive rotation being so chosen that ω is always positive. Put

$$r = (\nu/\omega)^{\frac12}\eta, \qquad z = (\nu/\omega)^{\frac12}\zeta, \qquad v_\theta = (\nu\omega)^{\frac12}\eta g(\zeta), \qquad v_z = (\nu\omega)^{\frac12}h(\zeta),$$

in which case equations (12) and (13) become

$$\tfrac14 h'^2 - \tfrac12 hh'' - g^2 = -\left(\frac{\omega^2+c}{\omega^2}\right) - \tfrac12 h''', \tag{15}$$

$$-gh' + g'h = g'', \tag{16}$$

where dashes denote differentiation with respect to ζ. The boundary conditions are now

$$h = h' = 0, \qquad g = 1, \quad \text{at} \quad \zeta = 0, \tag{17}$$

and, for the semi-infinite fluid (in which case $c = \gamma_1^2 - \omega^2$),

$$h' \to 0, \quad g \to \gamma_1/\omega, \quad \text{as} \quad \zeta \to \infty, \tag{18}$$

or, for the fluid between two rotating disks,

$$h = h' = 0, \qquad g = \gamma_2/\omega, \quad \text{at} \quad \zeta = \left(\frac{d^2\omega}{\nu}\right)^{\frac12}. \tag{19}$$

Numerical integration of equations (15) and (16) by the method used by Miss Hannah (**4**), and by Cochran (**6**) in the case of Kármán's problem, would probably be feasible, but tedious. Interest lies more in the general form of the solutions than in the numerical details so that only general observations about the streamlines in typical cases are presented here. The author has no evidence for the conclusions stated, other than that mentioned explicitly.

Streamlines of the flow bounded by a single disk

Consider first the family of solutions given by the boundary conditions (18). There is one member of this family for each value of γ_1/ω between $-\infty$ and $+\infty$, and for each member there is an appropriate value of V, the

axial velocity far from the disk. The members of the family may be divided into three classes within each of which the streamlines have much the same appearance.

Class (*a*). $+\infty > \gamma_1/\omega \geqslant 1$

The rotational velocity at the disk is, in this case, smaller than at any other point in the field so that the inward radial pressure gradient imposed

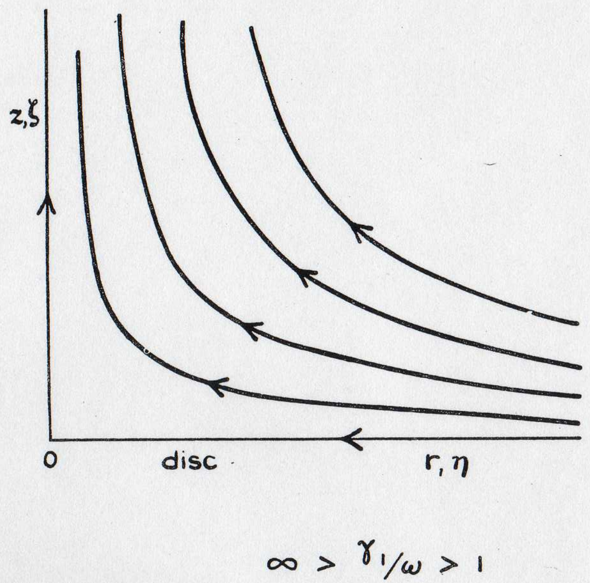


FIG. 1

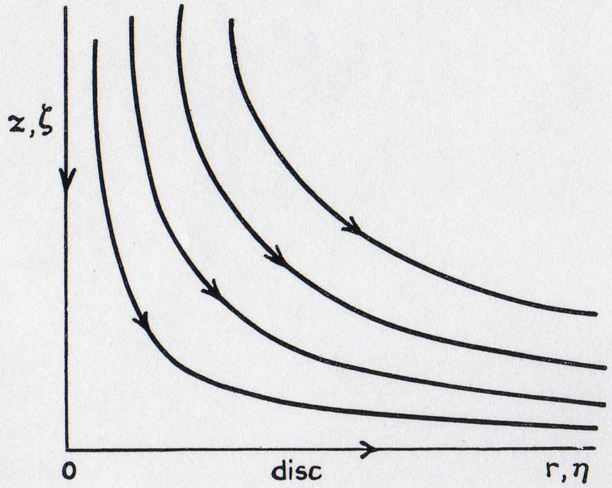
by the fluid at infinity is more than sufficient to keep the fluid near the disk moving in circles. Hence there is a radial flow inwards at points near the disk and an axial flow away from the disk (see Fig. 1; in this and other figures the streamlines refer, of course, to components of the motion in an axial plane only). At one end of the range $\gamma_1/\omega = \infty$, corresponding to a stationary disk and giving an approximation to the tea-cup flow, and at the other extreme $\gamma_1/\omega = 1$, corresponding to uniform rotation of the whole fluid with no axial motion.

Class (*b*). $1 \geqslant \gamma_1/\omega \geqslant 0$

The angular velocity at the disk is here a maximum and the rotation is everywhere in the same direction, so that the axial velocity is towards the disk (Fig. 2). The disk acts as a centrifugal fan, throwing fluid out radially and drawing it in axially. The extreme case $\gamma_1/\omega = 0$ is Kármán's problem. Note that the streamlines of members of this class are not obtained simply by reversing the streamlines of members of class (*a*).

Class (*c*). $0 > \gamma_1/\omega > -\infty$

The disk, and the fluid far from the disk, rotate in opposite directions, in this case so that at some value of ζ (i.e. of z), $g(\zeta) = 0$ and the fluid

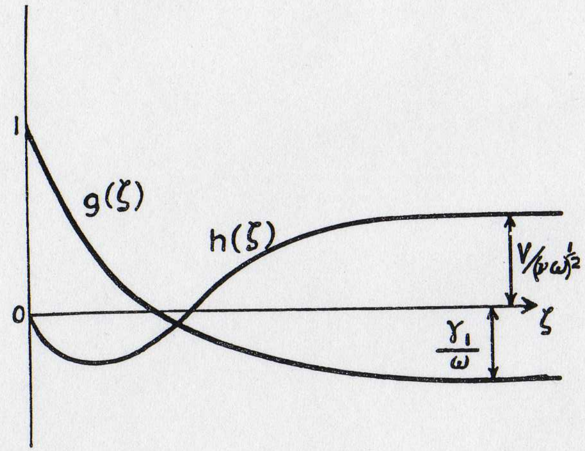


$1 > \gamma_1/\omega \geqslant 0$

Fig. 2

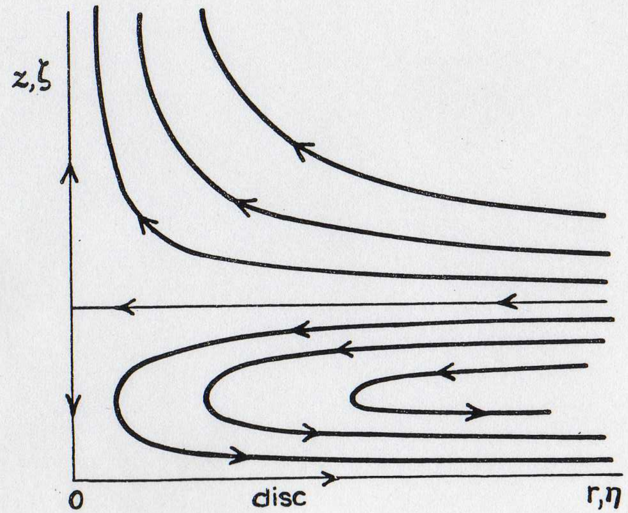
there has zero angular velocity about the axis $r = 0$. The angular velocity of the disk is thus greater in absolute magnitude than that of the fluid near it, and we anticipate that the radial velocity will be outward in the neighbourhood of the disk (i.e. the disk acts locally as a centrifugal fan) but inward elsewhere. This conclusion is supported by the following rough investigation of the form of the functions $g(\zeta)$ and $h(\zeta)$. At $\zeta = 0$ we have $g = 1$ and, it may be assumed, $g' < 0$. When ζ is large, g is asymptotic to the constant values γ_1/ω so that the function probably has the form shown in Fig. 3. Now equation (16) can be written in the form

$$h/g = -\int_0^{\zeta} \frac{g''}{g^2}\,d\zeta \quad \text{or} \quad = \frac{V}{(\nu\omega)^{\frac{1}{2}}}\Big/\frac{\gamma_1}{\omega} + \int_{\zeta}^{\infty} \frac{g''}{g^2}\,d\zeta, \tag{20}$$



$0 > \gamma_1/\omega > -\infty$

Fig. 3



$0 > \gamma_1/\omega > -\infty$

Fig. 4

whichever range avoids the singularity at the point where $g = 0$. On the supposition that $g'' > 0$ for all ζ, as in the sketch in Fig. 3, the variation of h required by these equations is sketched in Fig. 3 and the streamlines are shown in Fig. 4. Equation (16) shows that the value of ζ for which $h = 0$ must be at least as great as that for which $g = 0$. Apparently the critical plane on which the axial velocity h vanishes, which may not coincide with the plane of zero angular velocity, divides the flow field into two self-contained regions. The dividing plane coincides with the disk when $\gamma_1/\omega = -\infty$ and moves away to infinity as γ_1/ω increases from $-\infty$ to 0.

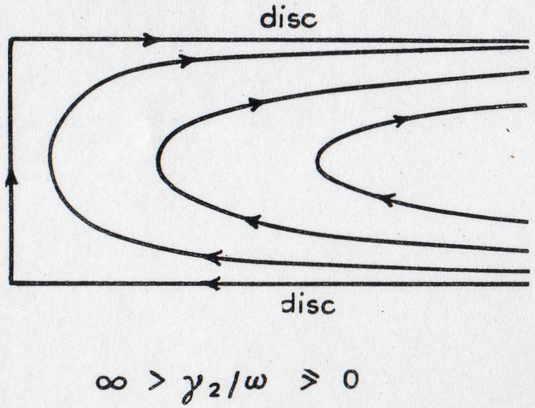


FIG. 5

Streamlines of the flow between two rotating disks

The streamlines in typical cases can again be sketched from elementary considerations. There is one solution for each value of the parameter γ_2/ω between $-\infty$ and $+\infty$ and for each value of the parameter $d^2\omega/\nu$ between 0 and $+\infty$. Variation of the parameter $d^2\omega/\nu$ has the effect of varying the extent of the region of rapid change of angular velocity, i.e. of controlling the boundary-layer character of the flow. The form of the streamlines does not vary radically with $d^2\omega/\nu$, so that the whole two-parameter family may be divided into two classes in which γ_2/ω takes opposite signs.

Class (*a*). $+\infty > \gamma_2/\omega \geqslant 0$

In this case the disks are rotating in the same direction and the magnitude of the angular velocity of the fluid varies monotonically with ζ. The radial velocity will be inwards near the slower rotating disk and outwards near the faster, which acts as a centrifugal fan, and the streamlines will be approximately as in Fig. 5. The extreme case $\gamma_2/\omega = 0$ has been

considered recently by Casal (**3**), who showed how the equations (15) and (16) can be integrated by expanding the velocities as power series in $d^2\omega/\nu$ which are convergent when $d^2\omega/\nu < 0{\cdot}17$.

An interesting situation arises when the Reynolds number $d^2\omega/\nu$ becomes very large. The effect of the no-slip condition is then confined to thin layers near each disk and the flow outside these layers is approximately as for a frictionless fluid, i.e. the angular velocity in this region is approximately independent of ζ. The streamlines are sketched in Fig. 6. The

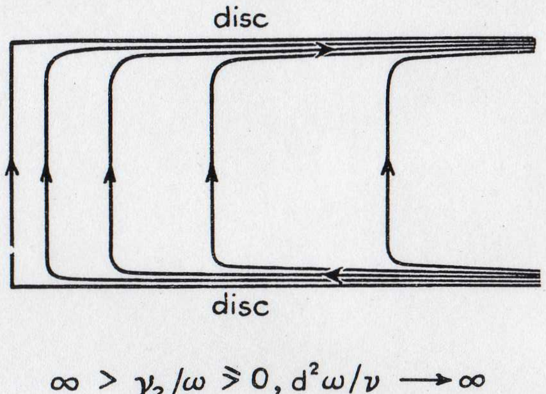


FIG. 6

uniform angular velocity in the interior of the fluid will presumably have a value such that the axial flow away from the slower rotating disk (considered as a disk rotating in a semi-infinite fluid which has constant angular velocity far from the disk) is just equal to the axial flow towards the faster rotating disk (considered in the same way). Thus if $V(\gamma_1/\omega)$ is the axial velocity far from a disk which has angular velocity ω in a semi-infinite fluid which has angular velocity γ_1 far from the disk, the angular velocity Ω in the region between the two disks at large Reynolds number is given by

$$V(\Omega/\omega) = -V(\Omega/\gamma_2). \tag{21}$$

The consideration of a single disk in a semi-infinite fluid showed that there will be a solution of this equation provided that, as already supposed, Ω is intermediate between ω and γ_2.

Class (*b*). $0 > \gamma_2/\omega > -\infty$

The disks now rotate in opposite directions and there is some plane between the disks on which the value of v_θ is zero. Thus the radial flow in the neighbourhood of each disk will be outward, with an inward radial flow in the interior of the fluid, as sketched in Fig. 7. As in the case of the semi-infinite fluid there will be a division of the flow into two self-contained regions, the dividing plane on which $v_z = 0$ being not necessarily identical (so far as can be seen without detailed numerical work) with the plane on which $v_\theta = 0$. The dividing plane coincides with the upper disk (angular

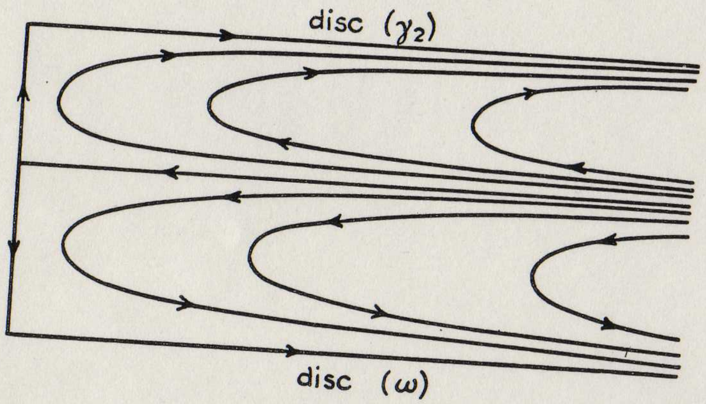


FIG. 7

velocity γ_2) when $\gamma_2/\omega = 0$ and moves down to the lower disk as γ_2/ω decreases from 0 to $-\infty$.

The case of very large Reynolds number is again very interesting. The effect of viscosity falls off rapidly with distance from each disk and outside thin layers near each disk the angular velocity is uniform and the stream-lines in a plane through the axis are axial. The uniform axial motion out-side the boundary layers will normally be from the slower to the faster rotating disk. However, the magnitude of the angular velocity decreases with distance from each disk and each disk must therefore act, locally at least, as a centrifugal fan. In the immediate neighbourhood of each disk the axial velocity must therefore be towards the disks, and the axial velocity must change sign somewhere between the disks. The means where-by this can happen have already been explored; the flow in the boundary layer near one disk (the slower) is presumably as sketched in Fig. 4, and near the other (the faster) as in Fig. 2. The streamlines thus have the

general shape shown in Fig. 8, which is drawn for the case $|\gamma_2| < |\omega|$. When $|\gamma_2|$ decreases to zero the outward radial flow near the upper disk vanishes and the flow reverts to that obtained by inverting Fig. 6. The angular velocity Ω in the region outside the two boundary layers is again determined by the equation

$$V(\Omega/\omega) = -V(\Omega/\gamma_2),$$

where V has the same meaning as in (21). Since Ω has the same sign

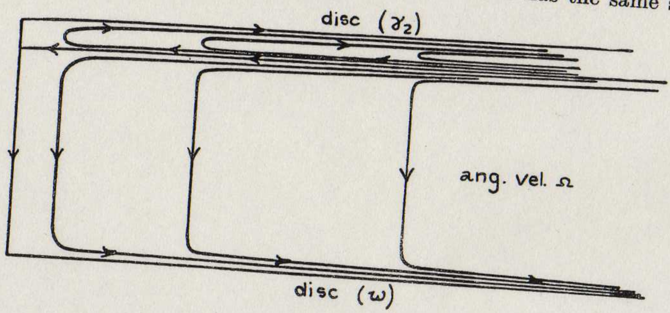


FIG. 8

as whichever of ω and γ_2 has the greater magnitude (so one believes in-tuitively), consistency of this equation for Ω with Figs. 1, 2, and 4 requires that

$$\text{if} \quad |\gamma_2/\omega| < 1, \quad \text{then} \quad 0 < \Omega/\omega < 1,$$

and reciprocally,

$$\text{if} \quad |\gamma_2/\omega| > 1, \quad \text{then} \quad 0 < \Omega/\gamma_2 < 1.$$

There does not appear to be any mathematical reason why there should not exist a solution for which the boundary layer on the faster rotating disk is the one in which the angular velocity changes direction. The axial velocity in the region outside the two boundary layers is in this case from the faster to the slower rotating disk. However, the solution would very probably be unstable, unlike that described above. The situation bears some resemblance to two-dimensional viscous flow through a diverging channel, and the analogy is closer if we imagine the two plane walls of the channel to be moving in their planes in the direction of flow at different speeds. There will be a region of reversed flow near one of the walls and two different solutions will be possible. But again it is probable that only the solution which gives reversed flow near the slower moving plane is stable.

The case $\gamma_2 = -\omega$ is singular, because there must then be a possible distribution of velocities symmetrical about the mid-plane (just as there is a singular symmetrical solution for flow in the diverging channel when the two planes have equal speeds—or in particular are stationary). Symmetry about the mid-plane implies that the boundary layers on each disk are mirror images of each other and that the axial velocity immediately

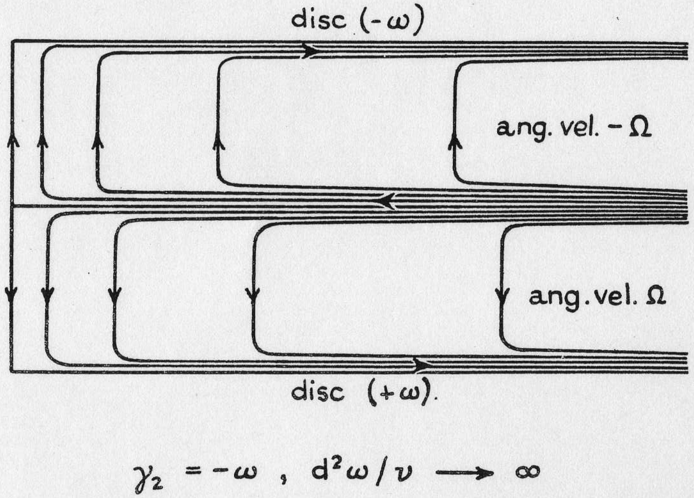


FIG. 9

outside each boundary layer is in each case towards the disk. Somewhere in the interior of the fluid the axial velocity must change appreciably and must reverse its direction. This cannot happen in the absence of a strong viscous effect, so that there is apparently a transition layer of rapid change as sketched in Fig. 9. The axial velocity and the angular velocity both change sign within this layer, while outside it (and outside the disk boundary layers) they are constant.

This singular solution may not be realizable experimentally, of course, but it has some intrinsic interest. For instance, it indicates that there is yet another solution of the type considered herein. The central transition layer is not directly connected with the two boundary layers (and in the limit of infinitely large Reynolds number, its position is arbitrary, provided it does not overlap with either of two of the boundary layers) and can be regarded as a transition region between two semi-infinite masses of fluid rotating with equal and opposite angular velocities $\mp\Omega$. Such a flow is

described by equations (15) and (16) (with $c = \Omega^2 - \omega^2$; note also that the reference angular velocity ω ought now to be replaced by Ω) with the new boundary conditions

$$h = h'' = 0, \quad g = 0, \quad \text{at} \quad \zeta = 0,$$

and

$$h' \to 0, \quad g \to \mp\Omega/\omega, \quad \text{as} \quad \zeta \to \mp\infty.$$

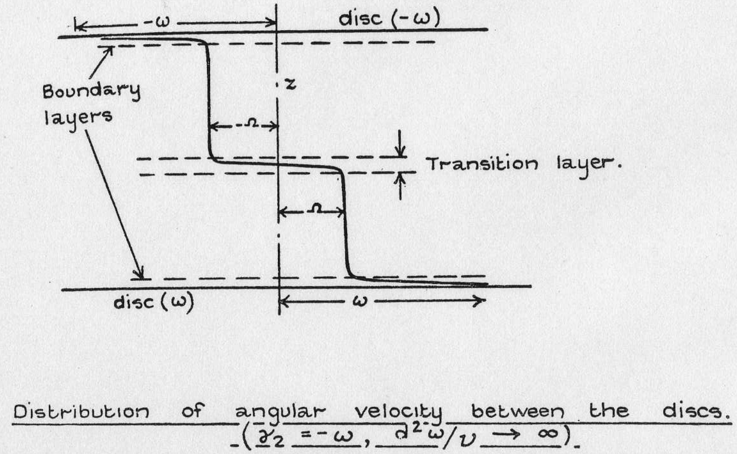


FIG. 10

If this solution gives an asymptotic axial flow $U(\Omega/\omega)$ away from the transition layer, the condition which determines the angular velocities $\mp\Omega$ in the non-viscous regions of the flow in Fig. 9 is

$$U(\Omega/\omega) = -V(\Omega/\omega),$$

which will have a solution provided $|\Omega| < |\omega|$. The angular velocity of the fluid between two disks evidently varies with z in the manner shown in Fig. 10.

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