$W(k)$: window function

$$g_j = \int_{-\infty}^{\infty} W(k)\cdot|k|\cdot\exp(2\pi i\cdot j\delta\cdot k)\,dk \;/\!/$$

$$h_j = \frac{i}{2\pi}\left\{\int_{-\infty}^{0} W(k)\cdot\exp(2\pi i\cdot j\delta\cdot k)\,dk - \int_{0}^{\infty} W(k)\exp(2\pi i\cdot j\delta\cdot k)\,dk\right\} \;/\!/$$

Ramachandran, $W(k) = \begin{cases} 1 & |k| \leq \frac{1}{2\delta} \\ 0 & \text{others} \end{cases}$

$$g_j = 2\int_{0}^{\frac{1}{2\delta}} k\cos(2\pi\cdot j\cdot\delta k)\,dk$$

$j=0$

$$g_0 = 2\int_0^{\frac{1}{2\delta}} k\,dk = \frac{1}{4\delta^2} \;/\!/$$

$j \neq 0$

$$g_j = 2\cdot\left(\frac{1}{2\pi\cdot j\cdot\delta}\right)^2\cdot\underbrace{\int_0^{\pi\cdot j}\theta\cdot\cos\theta\,d\theta}_{} \qquad \theta = 2\pi j\,\delta\cdot k$$

$$= [\cos\theta + \theta\sin\theta]_0^{\pi\cdot j} = (-1)^j - 1 = \begin{cases} 0 & j=\text{even} \\ -2 & \text{odd}\end{cases}$$

$$= \begin{cases} 0 & j=\text{even} \\ -\frac{1}{\pi^2\delta^2 j^2} & \text{odd} \end{cases} \;/\!/$$

$$h_j = \frac{i}{2\pi}\left\{\int_{-\frac{1}{2\delta}}^{0}\exp(2\pi i\cdot j\delta k)\,dk - \int_0^{\frac{1}{2\delta}}\exp(\ )\,dk\right\}$$

$$= \frac{i}{2\pi}\int_0^{\frac{1}{2\delta}}\{\cancel{\cos(2\pi j\delta k)} - i\sin() - \cancel{\cos()} - i\sin()\}\,dk$$

$$= \frac{1}{\pi}\int_0^{\frac{1}{2\delta}}\sin(2\pi j\delta k)\,dk$$

$j=0$のとき $h_0 = 0$

$$= \frac{1}{\pi}\frac{1}{2\pi j\delta}\int_0^{\pi j}\sin\theta\,d\theta = \frac{1-(-1)^j}{2\pi^2\cdot j\cdot\delta}$$

$$= \begin{cases} 0 & j=\text{even} \quad (j=0\text{を含む}) \\ \frac{1}{\pi^2\delta\cdot j} & \text{odd} \end{cases} \;/\!/$$

$$h_j = -\delta\cdot j\cdot g_j \;/\!/$$

Shepp. $W(k) = \begin{cases} \left|\sin\left(\frac{\pi}{2}\cdot\frac{k}{1/2\delta}\right)\Big/\left(\frac{\pi}{2}\cdot\frac{k}{1/2\delta}\right)\right| = \left|\frac{\sin(\pi\delta k)}{\pi\delta k}\right| & |k| \leq \frac{1}{2\delta} \\ 0 & \text{others} \end{cases}$

$$g_j = \int_{-\frac{1}{2\delta}}^{\frac{1}{2\delta}} \left|\frac{\sin(\pi\delta k)}{\pi\delta \cancel{k}}\right| \cdot \cancel{|k|} \cdot \exp(2\pi i\cdot j\delta\cdot k)\, dk$$

$$= \frac{2}{\pi\delta}\int_0^{\frac{1}{2\delta}} \sin(\pi\delta k)\cos(2\pi j\delta k)\, dk$$

$$= \frac{1}{2}\left\{\sin\pi\delta\cdot(1+2j)k + \sin\pi\delta\cdot(1-2j)k\right\}$$

$$= \frac{1}{\pi\delta}\left[-\frac{\cos\pi\delta\cdot(1+2j)\cdot k}{\pi\delta\cdot(1+2j)} - \frac{\cos\pi\delta\cdot(1-2j)\cdot k}{\pi\delta\cdot(1-2j)}\right]_0^{\frac{1}{2\delta}}$$

$$= \frac{1}{\pi^2\delta^2}\left(\frac{1}{1+2j} + \frac{1}{1-2j}\right)$$

($1\pm 2j$ は奇数なので $\cos\pi\delta\cdot(1\pm 2j)\cdot\frac{1}{2\delta} = \cos\frac{\pi}{2}(1\pm 2j) = 0$)

$$= \frac{1}{\pi^2\delta^2}\,\frac{2}{(1+2j)(1-2j)} = \frac{2}{\pi^2\cdot\delta^2\cdot(1-4j^2)} \quad //$$

$$h_j = \frac{i}{2\pi}\left\{\int_{-\frac{1}{2\delta}}^{0}\left|\frac{\sin(\pi\delta k)}{\pi\delta k}\right|\cdot\exp(2\pi i\cdot j\delta\cdot k)\,dk - \int_0^{\frac{1}{2\delta}}\left|\quad\right|\cdot\exp(\;)\,dk\right\}$$

$$= \frac{i}{2\pi^2\delta}\int_0^{\frac{1}{2\delta}}\frac{\sin(\pi\delta k)}{k}\cdot\left\{\cancel{\cos(2\pi j\delta k)} - i\cdot\sin(\;) - \cancel{\cos(\;)} - i\sin(\;)\right\}dk$$

$$= \frac{1}{\pi^2\delta}\int_0^{\frac{1}{2\delta}}\frac{\sin(\pi\delta k)\sin(2\pi j\delta k)}{k}\,dk$$

$\cos\theta = \cos|\theta|$ //

$$= \frac{1}{\pi^2\delta}\int_0^{\frac{1}{2\delta}}\frac{1}{2}\,\frac{\cos\pi\delta(1-2j)\cdot k - \cos\pi\delta(1+2j)\cdot k}{k}\,dk$$

$$= \frac{1}{2\pi^2\delta}\int_{\frac{\pi|1+2j|}{2}}^{\frac{\pi|1-2j|}{2}}\frac{\cos\theta}{\theta}\,d\theta \;\leftarrow\; \int_0^{t_2}\frac{\cos\theta}{\theta}d\theta - \int_0^{t_1}\frac{\cos\theta}{\theta}d\theta = \int_{t_1}^{t_2}\frac{\cos\theta}{\theta}d\theta$$

Cosine integral, $C_i(t) = -\int_t^{\infty}\frac{\cos\theta}{\theta}d\theta$,

$$\int_{t_1}^{t_2}\frac{\cos\theta}{\theta}d\theta = \int_{\infty}^{t_2}\frac{\cos\theta}{\theta}d\theta + \int_{t_1}^{\infty}\frac{\cos\theta}{\theta}d\theta = C_i(t_2) - C_i(t_1)$$

$$h_j = \frac{C_i\left(\left|\frac{\pi}{2}-\pi j\right|\right) - C_i\left(\left|\frac{\pi}{2}+\pi j\right|\right)}{2\pi^2\delta} \quad //$$

($j=0$ の場合も OK, $h_0 = 0$)

<- incorrect
see "ci.eqn"
or "ci.pdf"

Chesler, $W(k) = \begin{cases} \frac{1}{2}\left\{1+\cos\left(\pi \frac{k}{1/2\delta}\right)\right\} = \frac{1}{2}\left\{1+\cos(2\pi\delta k)\right\} & |k| \leqq \frac{1}{2\delta} \\ 0 & \text{others} \end{cases}$

$$g_j = \int_{-\frac{1}{2\delta}}^{\frac{1}{2\delta}} |k| \cdot \frac{1}{2}\left\{1+\cos(2\pi\delta k)\right\} \exp(2\pi i \cdot j\delta \cdot k)\, dk$$

$$= \int_0^{\frac{1}{2\delta}} k \cdot \left\{1+\cos(2\pi\delta k)\right\} \cos(2\pi \cdot j\delta \cdot k)\, dk$$

$$= \int_0^{\frac{1}{2\delta}} \left\{ k\cos(2\pi j\delta k) + k\underbrace{\cos(2\pi\delta k)\cos(2\pi j\delta k)}_{= \frac{1}{2}\left\{\cos 2\pi\delta(1+j)k + \cos 2\pi\delta\cdot(1-j)\cdot k\right\}} \right\} dk$$

$$g_0 = \int_0^{\frac{1}{2\delta}} \left\{ k + k\cos(2\pi\delta k)\right\} dk$$

$$= \frac{1}{2}\left[k^2\right]_0^{\frac{1}{2\delta}} + \frac{1}{(2\pi\delta)^2}\underbrace{\left[\cos\theta + \theta\sin\theta\right]_0^{\pi}}_{=-2} = \frac{1}{\cancel{2}\,8\,\delta^2} + \frac{(-2)}{(2\pi\delta)^2} = \frac{1}{2\delta^2}\left(\cancel{1}\,\tfrac{1}{4} - \frac{1}{\pi^2}\right) \quad //$$

$$g_{\pm 1} = \int_0^{\frac{1}{2\delta}} \left\{ k\cos(2\pi\delta k) + \frac{k}{2}\left[1+\cos(4\pi\delta k)\right]\right\} dk$$

$$= \frac{1}{(2\pi\delta)^2}\left[\cos\theta+\theta\sin\theta\right]_0^{\pi} + \frac{1}{4}\left[k^2\right]_0^{\frac{1}{2\delta}} + \frac{1}{2}\,\frac{1}{(4\pi\delta)^2}\underbrace{\left[\cos\theta+\theta\sin\theta\right]_0^{2\pi}}_{=0}$$

$$= \frac{-\cancel{2}}{\cancel{4}\pi^2\delta^2} + \frac{1}{4}\,\frac{1}{4\delta^2} = \frac{1}{2\delta^2}\left(\frac{1}{8} - \frac{1}{\pi^2}\right) \quad //$$

$|j| \geqq 2$

$$g_j = \frac{1}{(2\pi j\delta)^2}\left[\cos\theta+\theta\sin\theta\right]_0^{\pi j} + \frac{1}{2}\,\frac{\left[\cos\theta+\theta\sin\theta\right]^{\pi\cdot(j+1)}}{\left\{2\pi\delta\cdot(j+1)\right\}^2} + \frac{1}{2}\,\frac{\left[\cos\theta+\theta\sin\theta\right]^{\pi\cdot(j-1)}}{\left\{2\pi\delta\cdot(j-1)\right\}^2}$$

$$= \frac{(-1)^j - 1}{(2\pi j\delta)^2} + \frac{1}{2}\,\frac{(-1)^{j+1}-1}{\left\{2\pi(j+1)\delta\right\}^2} + \frac{1}{2}\,\frac{(-1)^{j-1}-1}{\left\{2\pi\cdot(j-1)\cdot\delta\right\}^2}$$

$j = \text{even}$

$$g_j = -\frac{1}{(2\pi\delta)^2}\cdot\left\{\frac{1}{(j+1)^2} + \frac{1}{(j-1)^2}\right\} = -\frac{j^2+1}{2\pi^2\cdot\delta^2\cdot(j^2-1)^2} \quad //$$

$j = \text{odd}$

$$g_j = -\frac{1}{2\pi^2\delta^2}\,\frac{1}{j^2} \quad //$$

$j=0$で $-\frac{1}{2\pi^2\delta^2}$

$g_0 = \frac{1}{2\delta^2} - \frac{1}{2\pi^2\delta^2}$ 一致しない

$j=\pm1$で $-\frac{1}{2\pi^2\delta^2}$, $g_{\pm1} = \frac{1}{16\delta^2} - \frac{1}{2\pi^2\delta^2}$ 一致しない.

$$h_j = \frac{i}{4\pi}\left\{ \int_{-\frac{1}{2\delta}}^{0} [1+\cos(2\pi\delta k)] \exp(2\pi i \cdot j\,\delta k)\, dk - \int_0^{\frac{1}{2\delta}} [\ ] \exp(\,)\, dk \right\}$$

$$= \frac{i}{4\pi} \int_0^{\frac{1}{2\delta}} \left\{ [\ ] \cdot \cancel{\cos(2\pi j \delta k)} - i \cdot [\ ] \sin(\,) - \cancel{[\ ]\cos(\,)} - i\,[\ ] \sin(\,) \right\} dk$$

$$= \frac{1}{2\pi} \int_0^{\frac{1}{2\delta}} \{ 1+\cos(2\pi\delta k) \} \sin(2\pi j \delta k)\, dk$$

$$\cos\alpha \sin\beta = \frac{1}{2}\{ \sin(\alpha+\beta) - \sin(\alpha-\beta) \}$$

$h_0 = 0$

$j \neq 0$

$$h_j = \frac{1}{2\pi} \int_0^{\frac{1}{2\delta}} \left\{ \sin(2\pi j \delta k) + \frac{1}{2} \sin 2\pi(1+j)\cdot\delta k - \frac{1}{2}\sin 2\pi\cdot(1-j)\cdot\delta k \right\} dk$$

$$h_{\pm 1} = \frac{1}{2\pi} \int_0^{\frac{1}{2\delta}} \left\{ \pm \sin(2\pi\delta k) \pm \frac{1}{2}\sin(4\pi\delta k) \right\} dk$$

$$= \frac{1}{2\pi}\left\{ \pm \frac{[-\cos\theta]_0^{\pi}}{2\pi\delta} \pm \cancel{\frac{[-\cos\theta]_0^{2\pi}}{2\cdot 4\pi\delta}} \right\} = \pm \frac{1}{2\pi^2\delta} \quad //$$

$|j| \geq 2$

$$h_j = \frac{1}{2\pi}\left\{ \frac{[-\cos\theta]_0^{\pi j}}{2\pi\delta\cdot j} + \frac{[-\cos\theta]_0^{\pi(1+j)}}{2\cdot 2\pi\delta\cdot(1+j)} - \frac{[-\cos\theta]_0^{\pi\cdot(1-j)}}{2\cdot 2\pi\delta\cdot(1-j)} \right\}$$

$$= -\frac{1}{4\pi^2\delta}\left\{ \frac{(-1)^j - 1}{j} + \frac{(-1)^{j+1}-1}{2\cdot(j+1)} + \frac{(-1)^{j-1}-1}{2\cdot(j-1)} \right\}$$

$j = \text{even}$

$$h_j = \frac{1}{4\pi^2\delta}\left(\frac{1}{j+1} + \frac{1}{j-1} \right) = \frac{j}{2\pi^2\delta\cdot(j^2-1)} \qquad \left(\begin{array}{c} j=0 \text{ の場合も OK} \\ h_0 = 0 \end{array} \right)$$

$j = \text{odd}$

$$h_j = \frac{1}{2\pi^2\delta\cdot j} \qquad \left(\begin{array}{c} j=\pm 1 \text{ の場合も OK} \\ h_{\pm 1} = \pm\frac{1}{2\pi^2\delta} \end{array} \right)$$

$\therefore$

$$h_j = \frac{1}{2\pi^2\delta} \times \begin{cases} \dfrac{j}{j^2-1} & j = \text{even} \ \ (j=0 \text{ を含む}) \\ \dfrac{1}{j} & \text{odd} \ \ (j=\pm 1 \text{ を含む}) \end{cases} \quad //$$

jが~~奇数~~のとき、$h_j = -\delta\cdot j\cdot g_j$ //