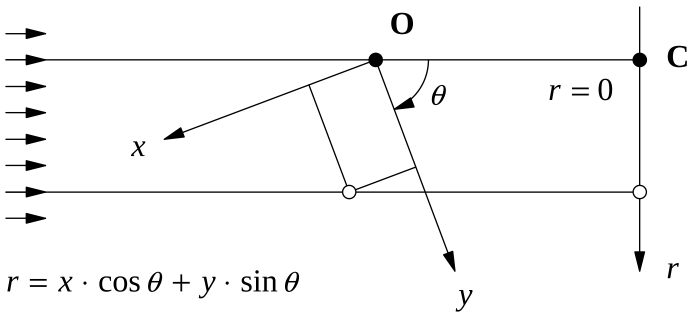
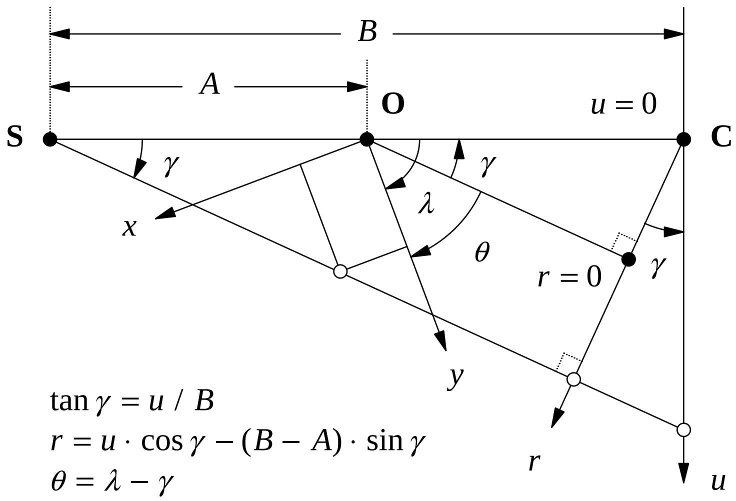
image reconstruction for fan beam CT

version 2 (2015/03/06, 2015/03/18, 2017/04/08)

parallel beam CT



fan beam CT



synthetic projection mehod

$p(r, \theta) = p_B(u, \lambda)$

where

$p_B(u, \lambda)$: observed projection of fan beam CT

$p(r, \theta)$: synthetic projection for the equivalent parallel beam CT

with

$$\left.\begin{aligned} r &= u \cdot \cos\gamma - (B - A) \cdot \sin\gamma = \frac{A \cdot u}{\sqrt{B^2 + u^2}} \\ \theta &= \lambda - \gamma = \lambda - \tan^{-1}\left(\frac{u}{B}\right) \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} u &= \frac{B \cdot r}{\sqrt{A^2 - r^2}} \\ \lambda &= \theta + \tan^{-1}\left(\frac{r}{\sqrt{A^2 - r^2}}\right) \end{aligned}\right.$$

detector interval

$$\delta_r = \frac{A}{B} \cdot \delta_u$$

convolution back-projection (CBP) method for fan beam CT

CBP method for parallel beam CT

$$\left.\begin{array}{ll}\text{reconstruction filter :} & g(r) = \int_{-\infty}^{+\infty} |k| \cdot \exp(2 \cdot \pi \cdot i \cdot r \cdot k) \ \ dk \\ \text{convolution :} & q(r,\theta) = \int_{-\infty}^{+\infty} g(r-\rho) \cdot \ p(\rho,\theta) \ \ d\rho \\ \text{back-projection :} & f(x,y) = \frac{1}{2} \cdot \int_0^{2\pi} q(x \cdot \cos\theta + y \cdot \sin\theta, \theta) \ \ d\theta \end{array}\right\}$$

$$\rightarrow \ f(x,y) = \frac{1}{2} \cdot \int_0^{2\pi} \int_{-\infty}^{+\infty} g(x \cdot \cos\theta + y \cdot \sin\theta - r) \cdot \ p(r,\theta) \ \ dr \ d\theta$$

change of variables of the double integral

$$\int\int h\big(r(u,\lambda), \theta(u,\lambda) \big) \ \ dr \ d\theta = \int\int h\big(r(u,\lambda), \theta(u,\lambda) \big) \cdot \| J \| \ \ du \ d\lambda$$

where $\| J \| \equiv \left| \frac{\partial r}{\partial u} \cdot \frac{\partial \theta}{\partial \lambda} - \frac{\partial r}{\partial \lambda} \cdot \frac{\partial \theta}{\partial u} \right| = \frac{A \cdot B^2}{(B^2+u^2)^{3/2}}$: absolute value of determinant for Jacobian matrix

argument of the reconstruction filter

$$\begin{aligned} x \cdot \cos\theta + y \cdot \sin\theta - r &= x \cdot \cos(\lambda-\gamma) + y \cdot \sin(\lambda-\gamma) - \big(u \cdot \cos\gamma - (B-A) \cdot \sin\gamma \big) \\ &= \frac{B \cdot (x \cdot \cos\lambda + y \cdot \sin\lambda) - u \cdot (A - x \cdot \sin\lambda + y \cdot \cos\lambda)}{\sqrt{B^2+u^2}} \\ &= \frac{A - x \cdot \sin\lambda + y \cdot \cos\lambda}{\sqrt{B^2+u^2}} \cdot \left(\frac{B \cdot (x \cdot \cos\lambda + y \cdot \sin\lambda)}{A - x \cdot \sin\lambda + y \cdot \cos\lambda} - u \right) \end{aligned}$$

notable property of the reconstruction filter

$$\begin{aligned} g(\alpha \cdot \beta) &= \int_{-\infty}^{+\infty} |k| \cdot \exp(2 \cdot \pi \cdot i \cdot \alpha \cdot \beta \cdot k) \ \ dk \\ &= \int_{-\infty}^{+\infty} \left| \frac{\kappa}{\alpha} \right| \cdot \exp(2 \cdot \pi \cdot i \cdot \beta \cdot \kappa) \cdot \frac{1}{\alpha} \ \ d\kappa = \frac{1}{\alpha^2} \cdot g(\beta) \end{aligned}$$

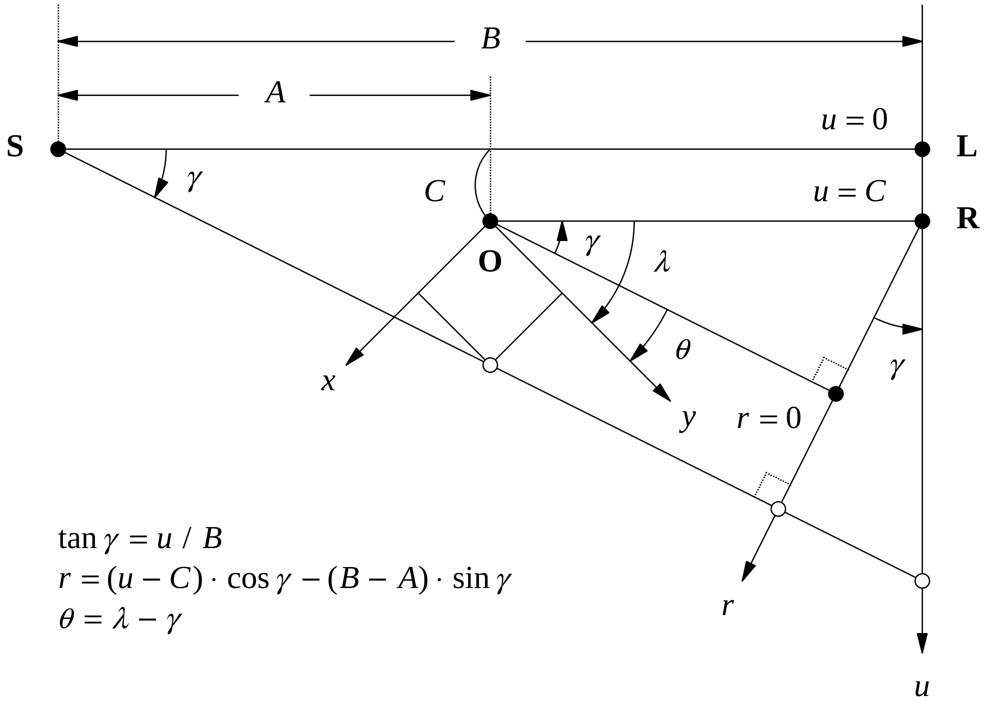
CBP method for fan beam CT

$$\begin{pmatrix} R(x,y,\lambda) \\ S(x,y,\lambda) \end{pmatrix} \equiv \begin{pmatrix} x \cdot \cos\lambda + y \cdot \sin\lambda \\ -x \cdot \sin\lambda + y \cdot \cos\lambda \end{pmatrix} = \begin{pmatrix} \cos\lambda, & \sin\lambda \\ -\sin\lambda, & \cos\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} f(x,y) &= \frac{1}{2} \cdot \int_0^{2\pi} \int_{-\infty}^{+\infty} g\left(\frac{A+S}{\sqrt{B^2+u^2}} \cdot \left(\frac{B \cdot R}{A+S} - u \right) \right) \cdot \ p_B(u,\lambda) \cdot \| J \| \ \ du \ d\lambda \\ &= \frac{1}{2} \cdot \int_0^{2\pi} \int_{-\infty}^{+\infty} \left(\frac{\sqrt{B^2+u^2}}{A+S} \right)^2 \cdot g\left(\frac{B \cdot R}{A+S} - u \right) \cdot \ p_B(u,\lambda) \cdot \frac{A \cdot B^2}{(B^2+u^2)^{3/2}} \ \ du \ d\lambda \end{aligned}$$

$$\rightarrow \left\{\begin{array}{ll}\text{reconstruction filter :} & g(u) = \int_{-\infty}^{+\infty} |k| \cdot \exp(2 \cdot \pi \cdot i \cdot u \cdot k) \ \ dk \\ \text{convolution :} & q_B(u,\lambda) = \int_{-\infty}^{+\infty} g(u-v) \cdot \frac{A}{\sqrt{B^2+v^2}} \cdot \ p_B(v,\lambda) \ \ dv \\ \text{back-projection :} & f(x,y) = \frac{1}{2} \cdot \int_0^{2\pi} \left(\frac{B}{A+S} \right)^2 \cdot q_B\left(\frac{B}{A+S} \cdot R, \lambda \right) \ d\lambda \end{array}\right.$$

when rotation axis is not on the line connecting light source and illumination center



$$p(r,\theta) = p_B(u,\lambda)$$

$$r = (u - C)\cdot\cos\gamma - (B - A)\cdot\sin\gamma = \frac{A\cdot u - B\cdot C}{\sqrt{B^2+u^2}}$$

$$\theta = \lambda - \gamma = \lambda - \tan^{-1}\left(\frac{u}{B}\right)$$

$$\left.\begin{array}{l} \dfrac{\partial r}{\partial u} = \dfrac{B\cdot(A\cdot B + C\cdot u)}{(B^2+u^2)^{3/2}} \\ \dfrac{\partial r}{\partial \lambda} = 0 \\ \dfrac{\partial \theta}{\partial u} = -\dfrac{B}{B^2+u^2} \\ \dfrac{\partial \theta}{\partial \lambda} = 1 \end{array}\right\} \rightarrow \|J\| \equiv \begin{vmatrix} \dfrac{\partial r}{\partial u}, & \dfrac{\partial r}{\partial \lambda} \\ \dfrac{\partial \theta}{\partial u}, & \dfrac{\partial \theta}{\partial \lambda} \end{vmatrix} = \frac{B\cdot(A\cdot B + C\cdot u)}{(B^2+u^2)^{3/2}}$$

$$\begin{aligned} x\cdot\cos\theta + y\cdot\sin\theta - r &= x\cdot\cos(\lambda-\gamma) + y\cdot\sin(\lambda-\gamma) - \big((u-C)\cdot\cos\gamma - (B-A)\cdot\sin\gamma\big) \\ &= \frac{B\cdot(C + x\cdot\cos\lambda + y\cdot\sin\lambda) - u\cdot(A - x\cdot\sin\lambda + y\cdot\cos\lambda)}{\sqrt{B^2+u^2}} \\ &= \frac{A - x\cdot\sin\lambda + y\cdot\cos\lambda}{\sqrt{B^2+u^2}}\cdot\left(\frac{B\cdot(C + x\cdot\cos\lambda + y\cdot\sin\lambda)}{A - x\cdot\sin\lambda + y\cdot\cos\lambda} - u\right) \end{aligned}$$

$$g(\alpha\cdot\beta) = \int_{-\infty}^{+\infty} |k|\cdot\exp(2\cdot\pi\cdot i\cdot\alpha\cdot\beta\cdot k)\ dk = \cdots = \frac{1}{\alpha^2}\cdot g(\beta)$$

$$\begin{pmatrix} R(x,\,y,\,\lambda) \\ S(x,\,y,\,\lambda) \end{pmatrix} \equiv \begin{pmatrix} x\cdot\cos\lambda + y\cdot\sin\lambda \\ -\,x\cdot\sin\lambda + y\cdot\cos\lambda \end{pmatrix} = \begin{pmatrix} \cos\lambda, & \sin\lambda \\ -\sin\lambda, & \cos\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} f(x,\,y) &= \tfrac{1}{2}\cdot\int_0^{2\pi}\int_{-\infty}^{+\infty} g(x\cdot\cos\theta + y\cdot\sin\theta - r)\cdot\, p(r,\,\theta)\;\; dr\; d\theta \\ &= \tfrac{1}{2}\cdot\int_0^{2\pi}\int_{-\infty}^{+\infty} g\left(\frac{A+S}{\sqrt{B^2+u^2}}\cdot\left(\frac{B\cdot(C+R)}{A+S} - u\right)\right)\cdot\, p_B(u,\,\lambda)\cdot\|J\|\;\; du\; d\lambda \\ &= \tfrac{1}{2}\cdot\int_0^{2\pi}\int_{-\infty}^{+\infty} \left(\frac{\sqrt{B^2+u^2}}{A+S}\right)^2\cdot g\left(\frac{B\cdot(C+R)}{A+S} - u\right)\cdot\, p_B(u,\,\lambda)\cdot\frac{B\cdot(A\cdot B + C\cdot u)}{(B^2+u^2)^{3/2}}\;\; du\; d\lambda \\ &= \tfrac{1}{2}\cdot\int_0^{2\pi}\int_{-\infty}^{+\infty} \left(\frac{B}{A+S}\right)^2\cdot g\left(\frac{B\cdot(C+R)}{A+S} - u\right)\cdot\, p_B(u,\,\lambda)\cdot\frac{A+\frac{C}{B}\cdot u}{\sqrt{B^2+u^2}}\;\; du\; d\lambda \end{aligned}$$

$$\rightarrow \begin{cases} \text{reconstruction filter :} & g(u) = \int_{-\infty}^{+\infty} |k|\cdot\exp(2\cdot\pi\cdot i\cdot u\cdot k)\;\; dk \\ \text{convolution :} & q_B(u,\,\lambda) = \int_{-\infty}^{+\infty} g(u-v)\cdot\frac{A+\frac{C}{B}\cdot v}{\sqrt{B^2+v^2}}\cdot p_B(v,\,\lambda)\;\; dv \\ \text{back-projection :} & f(x,\,y) = \tfrac{1}{2}\cdot\int_0^{2\pi}\left(\frac{B}{A+S}\right)^2\cdot q_B\left(\frac{B}{A+S}\cdot(C+R),\,\lambda\right)\; d\lambda \end{cases}$$

Feldkamp, Davis and Kress (FDK) method for cone beam CT

reconstruction filter

$$g(u) = \int_{-\infty}^{+\infty} |k|\cdot\exp(2\cdot\pi\cdot i\cdot u\cdot k)\;\; dk$$

convolution

$$q_B(u,\,w,\,\lambda) = \int_{-\infty}^{+\infty} g(u-v)\cdot\frac{A+\frac{C}{B}\cdot v}{\sqrt{B^2+v^2+w^2}}\cdot p_B(v,\,w,\,\lambda)\;\; dv$$

back-projection

$$f(x,\,y,\,z) \approx \tfrac{1}{2}\cdot\int_0^{2\pi}\left(\frac{B}{A+S}\right)^2\cdot q_B\left(\frac{B}{A+S}\cdot(C+R),\,\frac{B}{A+S}\cdot z,\,\lambda\right)\; d\lambda$$

where

$x = y = 0$ at the rotation axis, **O**
$u = w = 0$ at the illumination center, **L**
$z = 0$ at the intersection point of the rotation axis and the plane of $w = 0$