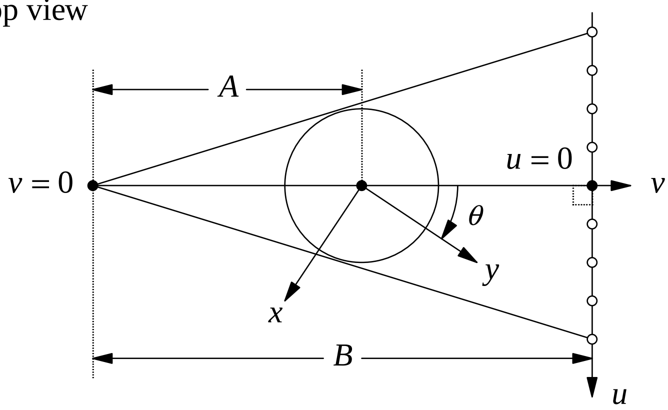
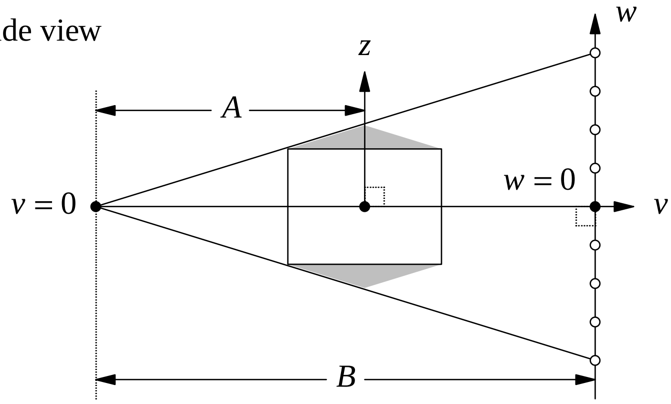
Feldkamp, Davis and Kress (FDK) method for cone beam CT

version 2 (2015/02/26, 2015/03/16, 2015/03/24)

top view



side view



Kak and Slaney (1988, page 104 - 107)

projection on the planes perpendicular to v-axis

$p_B(u, w, \theta)$: observed projection at $v = B$

$p_A(r, t, \theta) = p_B\left(\frac{B}{A} \cdot r, \frac{B}{A} \cdot t, \theta\right)$: equivalent projection at $v = A$

convolution (filtering)

$$q_A(r, t, \theta) = \int_{-\infty}^{+\infty} g(r - \rho) \cdot \frac{A \cdot p_A(\rho, t, \theta)}{\sqrt{A^2 + \rho^2 + t^2}} \, d\rho$$

where $g(r) = \int_{-\infty}^{+\infty} |\kappa| \cdot \exp(2 \cdot \pi \cdot i \cdot r \cdot \kappa) \, d\kappa$: reconstruction filter

back-projection

$$f(x, y, z) \approx \frac{1}{2} \cdot \int_0^{2\pi} \left(\frac{A}{A + S}\right)^2 \cdot q_A\left(\frac{A \cdot R}{A + S}, \frac{A \cdot z}{A + S}, \theta\right) d\theta$$

where $\begin{pmatrix} R(x, y, \theta) \\ S(x, y, \theta) \end{pmatrix} = \begin{pmatrix} \cos\theta, & \sin\theta \\ -\sin\theta, & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

equations for the FDK method

notable property of the reconstruction filter

$$g(\alpha \cdot \beta) = \int_{-\infty}^{+\infty} |\kappa| \cdot \exp(2 \cdot \pi \cdot i \cdot \alpha \cdot \beta \cdot \kappa)\ d\kappa$$

$$= \int_{-\infty}^{+\infty} \left| \frac{\kappa'}{\alpha} \right| \cdot \exp(2 \cdot \pi \cdot i \cdot \beta \cdot \kappa') \cdot \frac{1}{\alpha}\ d\kappa' = \frac{1}{\alpha^2} \cdot g(\beta)$$

filtered projection of $p_B(u, w, \theta)$

$$q_B(u, w, \theta) = \left(\frac{A}{B}\right)^2 \cdot q_A\left(\tfrac{A}{B} \cdot u,\ \tfrac{A}{B} \cdot w,\ \theta\right)$$

$$= \left(\frac{A}{B}\right)^2 \cdot \int_{-\infty}^{+\infty} g\left(\tfrac{A}{B} \cdot u - \rho\right) \cdot \frac{A \cdot p_A\left(\rho,\ \tfrac{A}{B} \cdot w,\ \theta\right)}{\sqrt{A^2 + \rho^2 + \left(\tfrac{A}{B} \cdot w\right)^2}}\ d\rho$$

$$= \left(\frac{A}{B}\right)^2 \cdot \int_{-\infty}^{+\infty} g\left(\tfrac{A}{B} \cdot (u - \upsilon)\right) \cdot \frac{A \cdot p_A\left(\tfrac{A}{B} \cdot \upsilon,\ \tfrac{A}{B} \cdot w,\ \theta\right)}{\sqrt{A^2 + \left(\tfrac{A}{B} \cdot \upsilon\right)^2 + \left(\tfrac{A}{B} \cdot w\right)^2}} \cdot \frac{A}{B}\ d\upsilon$$

$$= \left(\frac{A}{B}\right)^2 \cdot \int_{-\infty}^{+\infty} \left(\frac{B}{A}\right)^2 \cdot g(u - \upsilon) \cdot \frac{A \cdot p_B(\upsilon, w, \theta)}{\sqrt{B^2 + \upsilon^2 + w^2}}\ d\upsilon$$

$$= \int_{-\infty}^{+\infty} g(u - \upsilon) \cdot \frac{A \cdot p_B(\upsilon, w, \theta)}{\sqrt{B^2 + \upsilon^2 + w^2}}\ d\upsilon$$

back-projection using $q_B(u, w, \theta)$

$$f(x, y, z) \approx \frac{1}{2} \cdot \int_0^{2\pi} \left(\frac{A}{A+S}\right)^2 \cdot \left(\frac{B}{A}\right)^2 \cdot q_B\left(\frac{B}{A} \cdot \frac{A \cdot R}{A+S},\ \frac{B}{A} \cdot \frac{A \cdot z}{A+S},\ \theta\right) d\theta$$

$$= \frac{1}{2} \cdot \int_0^{2\pi} \left(\frac{B}{A+S}\right)^2 \cdot q_B\left(\frac{B \cdot R}{A+S},\ \frac{B \cdot z}{A+S},\ \theta\right) d\theta$$

implementation of the FDK method

given parameters and data

parameters

A and B	: distance from source to rotation axis and to detector array
δ_u and δ_w	: interval of detectors along u- and w-direction
N_u and N_w	: number of detectors along u- and w-direction
O_u and O_w	: center position of detector array along u- and w-direction
θ_0 and N_θ	: offset angle and total angular steps of the sample rotation

data

$$P_{j,l,m} = p_B(u_j, w_l, \theta_m)$$

where

$$u_j = \delta_u \cdot (j - O_u) \quad \text{for } j = 0, \cdots, N_u - 1$$

$$w_l = \delta_w \cdot (l - O_w) \quad \text{for } l = 0, \cdots, N_w - 1$$

$$\theta_m = \theta_0 + 2 \cdot \pi \cdot m \,/\, N_\theta \quad \text{for } m = 0, \cdots, N_\theta - 1$$

numerical integration

value of the reconstruction filter under the Nyquist frequency

$$
\begin{aligned}
G_j &= g(\delta_u \cdot j) \\
&\approx \int_{-1/(2\cdot\delta_u)}^{+1/(2\cdot\delta_u)} |\kappa| \cdot \exp(2\cdot\pi\cdot i\cdot\delta_u\cdot j\cdot\kappa)\ \ d\kappa \\
&= \frac{1}{\delta_u^2}\cdot\begin{cases} 1/4 & \text{for } \ j=0 \\ -1/(\pi\cdot j)^2 & \text{for } \ j=\text{odd number} \\ 0 & \text{for } \ j=\text{even number} \end{cases}
\end{aligned}
$$

convolution

$$
\begin{aligned}
Q_{j,\,l,\,m} &= q_B(u_j,\, w_l,\, \theta_m) \\
&\approx \delta_u \cdot \sum_{k=0}^{N_u-1} g(u_j - \upsilon_k)\cdot\frac{A\cdot p_B(\upsilon_k,\, w_l,\, \theta_m)}{\sqrt{B^2+\upsilon_k^2+w_l^2}} \\
&= \delta_u \cdot \sum_{k=0}^{N_u-1} G_{j-k}\cdot\frac{A\cdot P_{k,\,l,\,m}}{\sqrt{B^2+\upsilon_k^2+w_l^2}}
\end{aligned}
$$

back-projection

$$
\begin{aligned}
f(x,\,y,\,z) &\approx \tfrac{1}{2}\cdot\frac{2\cdot\pi}{N_\theta}\cdot\sum_{m=0}^{N_\theta-1}\left(\frac{B}{A+S_m}\right)^2\cdot q_B\!\left(\frac{B\cdot R_m}{A+S_m},\ \frac{B\cdot z}{A+S_m},\ \theta_m\right) \\
&\approx \frac{\pi}{N_\theta}\cdot\sum_{m=0}^{N_\theta-1}\left(\frac{B}{A+S_m}\right)^2\cdot I_m\!\left(\frac{B\cdot R_m}{A+S_m},\ \frac{B\cdot z}{A+S_m}\right)
\end{aligned}
$$

where $I_m(u,\, w)$: interpolated value of $q_B(u,\, w,\, \theta_m)$ using $Q_{j,\,l,\,m}$

$$
\begin{pmatrix} R_m \\ S_m \end{pmatrix} = \begin{pmatrix} R(x,\,y,\,\theta_m) \\ S(x,\,y,\,\theta_m) \end{pmatrix} = \begin{pmatrix} \cos\theta_m, & \sin\theta_m \\ -\sin\theta_m, & \cos\theta_m \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}
$$

reconstructed image of $f(x,\,y,\,z)$

imaging area

$$
x^2+y^2 \le \begin{cases}
\left(A - B\cdot\dfrac{z}{w_0}\right)^2 & \text{for } \ \dfrac{A}{B}\cdot w_0 \le z \le \dfrac{A-C}{B}\cdot w_0 & \text{: cone ▼} \\
C^2 & \text{for } \ \dfrac{A-C}{B}\cdot w_0 \le z \le \dfrac{A-C}{B}\cdot w_{N_w-1} & \text{: cylinder} \\
\left(A - B\cdot\dfrac{z}{w_{N_w-1}}\right)^2 & \text{for } \ \dfrac{A-C}{B}\cdot w_{N_w-1} \le z \le \dfrac{A}{B}\cdot w_{N_w-1} & \text{: cone ▲}
\end{cases}
$$

where $C = \min\left(-\dfrac{A\cdot u_0}{\sqrt{B^2+u_0^2}},\ \dfrac{A\cdot u_{N_u-1}}{\sqrt{B^2+u_{N_u-1}^2}}\right)$

pixel size (exactly, interval of grid points) along x-, y- and z-direction

$$
\begin{pmatrix} \delta_x \ \text{ and } \ \delta_y \\ \delta_z \end{pmatrix} = \frac{A}{B}\cdot\begin{pmatrix} \delta_u \\ \delta_w \end{pmatrix}
$$

calculation technique

convolution by the discrete Fourier transform (DFT)

when

$$G'_j = G_{j-N_u} \quad \text{for} \quad j = 0, \cdots, 2 \cdot N_u - 1$$

$$H'_j = \begin{cases} 0 & \text{for} \quad j = 0, \cdots, N_u - 1 \\ H_{j-N_u} & \text{for} \quad j = N_u, \cdots, 2 \cdot N_u - 1 \end{cases}$$

and

$$\begin{pmatrix} \hat{G}_k \\ \hat{H}_k \end{pmatrix} = \sum_{j=0}^{2 \cdot N_u - 1} \begin{pmatrix} G'_j \\ H'_j \end{pmatrix} \cdot \exp\left(-2 \cdot \pi \cdot i \cdot \frac{k \cdot j}{2 \cdot N_u}\right) \quad \text{for} \quad k = 0, \cdots, 2 \cdot N_u - 1$$

then

$$\sum_{k=0}^{N_u - 1} G_{j-k} \cdot H_k = \frac{1}{2 \cdot N_u} \cdot \sum_{k=0}^{2 \cdot N_u - 1} \hat{G}_k \cdot \hat{H}_k \cdot \exp\left(2 \cdot \pi \cdot i \cdot \frac{j \cdot k}{2 \cdot N_u}\right) \quad \text{for} \quad j = 0, \cdots, N_u - 1$$

bi-linear interpolation of $q_B(u, w, \theta_m)$ using $Q_{j,l,m}$

when

$$j = \text{floor}(O_u + u / \delta_u)$$
$$l = \text{floor}(O_w + w / \delta_w)$$

and

$$\varepsilon_u = O_u + u / \delta_u - j$$
$$\varepsilon_w = O_w + w / \delta_w - l$$

then

$$\begin{aligned} I_m(u, w) = Q_{j,l,m} &+ (Q_{j+1,l,m} - Q_{j,l,m}) \cdot \varepsilon_u \\ &+ (Q_{j,l+1,m} - Q_{j,l,m}) \cdot \varepsilon_w \\ &+ (Q_{j+1,l+1,m} - Q_{j,l+1,m} - Q_{j+1,l,m} + Q_{j,l,m}) \cdot \varepsilon_u \cdot \varepsilon_w \end{aligned}$$

calculation procedure

$F_{x,y,z}$: computer memory for the reconstructed value of $f(x, y, z)$

FDK method :

read parameters

calculate constant values and allocate computer memory

do $F_{x,y,z} = 0$ for each (x, y, z)

do the followings for each m :

read $P_{j,l,m}$ for each (j, l)

calculate $Q_{j,l,m}$ for each (j, l) by the fast Fourier transform (FFT)

do

$$F_{x,y,z} += \left(\frac{\pi}{N_\theta} \cdot \left(\frac{B}{A + S_m} \right)^2 \cdot I_m\left(\frac{B \cdot R_m}{A + S_m}, \ \frac{B \cdot z}{A + S_m} \right) \right)$$

for each (x, y, z) inside the imaging area