値１の球の像に Gaussian filter を施した像の値の空間分布、$I(r)$

r　球の中心からの距離

R　球の半径 (radius of sphere)

L　３次元 Gaussian filter のパラメータ (Gaussian radius)

$$f(a,b,c) = \begin{cases} 0 & \text{when } a^2+b^2+c^2 > R^2 \\ 1 & \text{when } a^2+b^2+c^2 \le R^2 \end{cases}$$

$$g(l) = \frac{1}{(\sqrt{\pi}\cdot L)^3}\cdot \exp\left(-\frac{l^2}{L^2}\right)$$

$$h(x,y,z) = \int_{-\infty}^{+\infty}\!\!\underset{a}{}\int_{-\infty}^{+\infty}\!\!\underset{b}{}\int_{-\infty}^{+\infty}\!\!\underset{c}{} f(a,b,c)\cdot g\left(\sqrt{(x-a)^2+(y-b)^2+(z-c)^2}\right)\cdot da\cdot db\cdot dc$$

$$\begin{aligned}
I(r) &\equiv h(0,0,r) \\
&= \int_{-\infty}^{+\infty}\!\!\underset{a}{}\int_{-\infty}^{+\infty}\!\!\underset{b}{}\int_{-\infty}^{+\infty}\!\!\underset{c}{} f(a,b,c)\cdot g\left(\sqrt{a^2+b^2+(r-c)^2}\right)\cdot da\cdot db\cdot dc \\
&= \int_0^{2\cdot\pi}\!\!\underset{\lambda}{}\int_0^{\pi}\!\!\underset{\theta}{}\int_0^{R}\!\!\underset{s}{} g\left(\sqrt{(s\cdot\cos\lambda\cdot\sin\theta)^2+(s\cdot\sin\lambda\cdot\sin\theta)^2+(r-s\cdot\cos\theta)^2}\right)\cdot s\cdot\sin\theta\cdot d\lambda\cdot s\cdot d\theta\cdot ds \\
&= 2\cdot\pi\cdot\int_0^{\pi}\!\!\underset{\theta}{}\int_0^{R}\!\!\underset{s}{} g\left(\sqrt{r^2+s^2-2\cdot r\cdot s\cdot\cos\theta}\right)\cdot s^2\cdot\sin\theta\cdot d\theta\cdot ds \\
&= \frac{2}{\sqrt{\pi}}\cdot\frac{1}{L^3}\cdot\int_0^{\pi}\!\!\underset{\theta}{}\int_0^{R}\!\!\underset{s}{} \exp\left(-\frac{r^2+s^2-2\cdot r\cdot s\cdot\cos\theta}{L^2}\right)\cdot s^2\cdot\sin\theta\cdot d\theta\cdot ds \\
&= \frac{2}{\sqrt{\pi}}\cdot\frac{1}{L^3}\cdot\int_0^{R}\!\!\underset{s}{} s^2\cdot\exp\left(-\frac{r^2+s^2}{L^2}\right)\cdot\int_0^{\pi}\!\!\underset{\theta}{}\exp\left(\frac{2\cdot r\cdot s\cdot\cos\theta}{L^2}\right)\cdot\sin\theta\cdot d\theta\cdot ds \\
&= \frac{2}{\sqrt{\pi}}\cdot\frac{1}{L^3}\cdot\int_0^{R}\!\!\underset{s}{} s^2\cdot\exp\left(-\frac{r^2+s^2}{L^2}\right)\cdot\frac{L^2}{r\cdot s}\cdot\sinh\left(\frac{2\cdot r\cdot s}{L^2}\right)\cdot ds \\
&= \frac{1}{\sqrt{\pi}}\cdot\frac{1}{L}\cdot\frac{1}{r}\cdot\int_0^{R}\!\!\underset{s}{} s\cdot\left(\exp\left(-\frac{(s-r)^2}{L^2}\right)-\exp\left(-\frac{(s+r)^2}{L^2}\right)\right)\cdot ds \\
&= \frac{1}{2}\cdot\left[+\operatorname{erf}(t)-\frac{1}{\sqrt{\pi}}\cdot\frac{L}{r}\cdot\exp(-t^2)\right]_{t=-\frac{r}{L}}^{t=\frac{R-r}{L}} - \frac{1}{2}\cdot\left[-\operatorname{erf}(t)-\frac{1}{\sqrt{\pi}}\cdot\frac{L}{r}\cdot\exp(-t^2)\right]_{t=+\frac{r}{L}}^{t=\frac{R+r}{L}} \\
&= \frac{1}{2}\cdot\left(\operatorname{erf}(m)+\operatorname{erf}(p)-\frac{1}{\sqrt{\pi}}\cdot\frac{L}{r}\cdot\left(\exp(-m^2)-\exp(-p^2)\right)\right) \quad \text{where} \begin{cases} m=(R-r)/L \\ p=(R+r)/L \end{cases}
\end{aligned}$$

$$\begin{aligned}
\lim_{r\to 0} I(r) &= \operatorname{erf}\left(\frac{R}{L}\right)-\frac{1}{2}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{L}{r}\cdot\exp\left(-\frac{R^2}{L^2}\right)\cdot\frac{4\cdot R\cdot r}{L^2} \\
&= \operatorname{erf}\left(\frac{R}{L}\right)-\frac{2}{\sqrt{\pi}}\cdot\frac{R}{L}\cdot\exp\left(-\frac{R^2}{L^2}\right)
\end{aligned}$$

$$\operatorname{erf}(s) \equiv \frac{2}{\sqrt{\pi}}\cdot\int_0^{s}\!\!\underset{t}{}\exp(-t^2)\cdot dt$$