値１の円の像に Gaussian filter を施した像の値の空間分布、$I(r)$

- r　円の中心からの距離
- R　円の半径 (radius of circle)
- L　２次元 Gaussian filter のパラメータ (Gaussian radius)

$$f(a,b)=\begin{cases} 0 & \text{when } a^2+b^2>R^2 \\ 1 & \text{when } a^2+b^2\le R^2 \end{cases}$$

$$g(l)=\frac{1}{\pi\cdot L^2}\cdot\exp\left(-\frac{l^2}{L^2}\right)$$

$$h(x,y)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(a,b)\cdot g\left(\sqrt{(x-a)^2+(y-b)^2}\right)\cdot da\cdot db$$

$$\begin{aligned}
I(r) &\equiv h(r,0)\\
&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(a,b)\cdot g\left(\sqrt{(r-a)^2+b^2}\right)\cdot da\cdot db\\
&=\int_0^{2\cdot\pi}\int_0^{R} g\left(\sqrt{(r-s\cdot\cos\theta)^2+(s\cdot\sin\theta)^2}\right)\cdot s\cdot d\theta\cdot ds\\
&=\frac{1}{\pi}\cdot\frac{1}{L^2}\cdot\int_0^{2\cdot\pi}\int_0^{R}\exp\left(-\frac{r^2+s^2-2\cdot r\cdot s\cdot\cos\theta}{L^2}\right)\cdot s\cdot d\theta\cdot ds\\
&=\frac{1}{\pi}\cdot\frac{1}{L^2}\cdot\int_0^{R} s\cdot\exp\left(-\frac{r^2+s^2}{L^2}\right)\cdot\int_0^{2\cdot\pi}\exp\left(\frac{2\cdot r\cdot s\cdot\cos\theta}{L^2}\right)\cdot d\theta\cdot ds\\
&=\frac{2}{L^2}\cdot\int_0^{R} s\cdot\exp\left(-\frac{r^2+s^2}{L^2}\right)\cdot \mathrm{I}_0\left(\frac{2\cdot r\cdot s}{L^2}\right)\cdot ds\\
&=\int_0^{\sqrt{2}\cdot\frac{R}{L}} t\cdot\exp\left(-\frac{1}{2}\cdot\left(\left(\sqrt{2}\cdot\frac{r}{L}\right)^2+t^2\right)\right)\cdot \mathrm{I}_0\left(\sqrt{2}\cdot\frac{r}{L}\cdot t\right)\cdot dt
\end{aligned}$$

Marcum Q-function (http://www.phaselockedsystems.com/NoncentralChiSquared.pdf)

$$\begin{aligned}
\mathrm{Q}(\alpha,\beta) &=\int_{\beta}^{\infty} t\cdot\exp\left(-\frac{1}{2}\cdot\left(t^2+\alpha^2\right)\right)\cdot \mathrm{I}_0(\alpha\cdot t)\cdot dt\\
&=\begin{cases} \exp\left(-\frac{1}{2}\cdot\left(\alpha^2+\beta^2\right)\right)\cdot\sum_{n=0}^{\infty}\left(\frac{\alpha}{\beta}\right)^n\cdot \mathrm{I}_n(\alpha\cdot\beta) & \text{when } \alpha<\beta \\ 1-\exp\left(-\frac{1}{2}\cdot\left(\alpha^2+\beta^2\right)\right)\cdot\sum_{n=1}^{\infty}\left(\frac{\beta}{\alpha}\right)^n\cdot \mathrm{I}_n(\alpha\cdot\beta) & \text{when } \alpha\ge\beta \end{cases}
\end{aligned}$$

$$Q(\alpha,0)=1$$

$$\begin{aligned}
I(r) &=\mathrm{Q}\left(\sqrt{2}\cdot\frac{r}{L},\ 0\right)-\mathrm{Q}\left(\sqrt{2}\cdot\frac{r}{L},\ \sqrt{2}\cdot\frac{R}{L}\right)\\
&=\begin{cases} 1-\exp\left(-\left(\frac{r-R}{L}\right)^2\right)\cdot\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n\cdot\exp\left(-\frac{2\cdot r\cdot R}{L^2}\right)\cdot \mathrm{I}_n\left(\frac{2\cdot r\cdot R}{L^2}\right) & \text{when } r<R \\ \exp\left(-\left(\frac{r-R}{L}\right)^2\right)\cdot\sum_{n=1}^{\infty}\left(\frac{R}{r}\right)^n\cdot\exp\left(-\frac{2\cdot r\cdot R}{L^2}\right)\cdot \mathrm{I}_n\left(\frac{2\cdot r\cdot R}{L^2}\right) & \text{when } r\ge R \end{cases}
\end{aligned}$$