

Gravitational potential and energy of homogeneous rectangular parallelepiped

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Gravitational potential and gravitational energy are presented in analytical form for homogeneous right parallelepiped.

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I. INTRODUCTION: BASIC FORMULAE

As it is well known, Newtonian gravitational potential, of homogeneous body with constant density ρ , at point (X, Y, Z) is defined as triple integral over the body's volume:

$$U(X, Y, Z) = G\rho \int \int \int u(X, Y, Z, x, y, z) dx dy dz \quad (1)$$

with

$$u(X, Y, Z, x, y, z) = [(x - X)^2 + (y - Y)^2 + (z - Z)^2]^{-1/2}; \quad (2)$$

here G stands for Newtonian constant of gravitation. In spite of almost 400-year-long attempts since Isaac Newton's times, the integral (1) is known in closed form (not in the series!) in quite a few cases [1]:

a) a piece of straight line, b) a sphere, c) an ellipsoid;

note that both cases a) and b) may be considered as particular cases of c). As to serial solution, the integral (1) is expressed in terms of the various kinds of series for external points outside the minimal sphere containing the whole body (non-necessary homogeneous), as well as for inner points close to the origin of co-ordinates. However in this note we do not touch the problem of serial solution and are only interested in *exact analytical* solution of (1).

Recently Kondrat'ev and Antonov [2] have obtained the rather sophisticated analytical formulas for the gravitational potential (and the gravitational energy) of some axial-symmetric figures, namely homogeneous lenses with spherical surfaces of different radii. In forthcoming paper [3] we present some new solutions for homogeneous bodies of revolution. Here we present the solution of Eq. (1) for homogeneous right (=rectangular) parallelepiped, (RP). The great simplification available in this case is that boundaries of integration over each of variables x, y, z are *fixed and thus independent on other two variables* [4]. This allows to get results in the closed form in terms of elementary functions. To this end we widely used Mathematica [5].

Let RP with center at the origin of Cartesian co-ordinates have lengths of sides $2a, 2b, 2c$, so that inside the RP, $-a \leq x \leq a, -b \leq y \leq b, -c \leq z \leq c$. Then Eq. (1) may be rewritten in the following symmetrical form:

$$U(X, Y, Z) = G\rho \int_{xm}^{xp} dx \int_{ym}^{yp} dy \int_{zm}^{zp} u(x, y, z) dz, \quad (3)$$

$$u(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}; \quad (4)$$

$$xm = -a - X, xp = a - X, ym = -b - Y, yp = a - Y, zm = -c - Z, zp = c - Z. \quad (5)$$

Note that now the boundaries (5) of integration in Eq.(3) depend on X, Y, Z , but this is not so "dangerous" as the case of dependence of boundaries on other variables of integration.

Now we are ready to start calculation of integral (3) in analytical form. Certainly final result should not depend on the order of integration [3] and this may be used to check the calculation.

First, an integration over z gives:

$$uz(X, Y, Z, x, y) = \int_{zm}^{zp} u(x, y, z) dz = [\ln(z + 1/u)]_{z=zm}^{z=zp}. \quad (6)$$

Here u is the same function as in Eq. (4).

Second, an integration over y gives:

$$uyz(X, Y, Z, x) = \left[\left[-x \arctan \frac{yzu}{x} + y \ln(z + 1/u) + z \ln(y + 1/u) \right]_{z=zm}^{z=zp} \right]_{y=ym}^{y=yp}. \quad (7)$$

And integration over x gives the final expression for the gravitational potential of homogeneous RP:

$$U(X, Y, Z) = G\rho \left[\left[\left[v(x, y, z) + v(y, z, x) + v(z, x, y) \right]_{z=zm}^{z=zp} \right]_{y=ym}^{y=yp} \right]_{x=xm}^{x=xp}; \quad (8)$$

$$v(x', y', z') = y'z' \ln[x' + (x'^2 + y'^2 + z'^2)^{1/2}] - \frac{x'^2}{2} \arctan \frac{y'z'}{x'(x'^2 + y'^2 + z'^2)^{1/2}}. \quad (9)$$

Here co-ordinates X, Y, Z are as given in (5). In Appendix we present the final result (8,9,5) in explicit form.

These formulas (8,9,5) may look cumbersome and unpractical while in fact they allow to easily (and exactly!) calculate gravitational potential of homogeneous RP at *arbitrary point inside* as well *outside* the body. The payment for this universality does not seem very high. Also we should note that obtaining the result (8,9,5) is non-trivial in the sense that direct using of Mathematica's command for integration of (1,2), `Integrate[f[x],x]`, (and especially command for definite integral, `Integrate[f[x],{x,xm,xp}]`), gives enormous complex expressions even leaving some integrals uncalculated!

Note that a physically evident symmetry relative to all three sign changes, $X \rightarrow -X$, $Y \rightarrow -Y$, and $Z \rightarrow -Z$ may be used to check final formulas.

Let us consider now the various particular cases.

II. A PIECE OF STRAIGHT LINE

This is one of a few known classical results. From Eq.(6), the 3D-potential, at arbitrary point (X, Y, Z) , of a piece of straight line, PSL, with constant linear density ρ and length $2c$, with center at origin of co-ordinates, PSL laying along the Z -axis, is as follows:

$$U_{line}(X, Y, Z) = G\rho \ln \left[\frac{c - Z + (X^2 + Y^2 + (c - Z)^2)^{1/2}}{-c - Z + (X^2 + Y^2 + (c + Z)^2)^{1/2}} \right]. \quad (10)$$

The PSL case may be considered as RP with two infinitesimal dimensions dx, dy . There is a scaling invariance: if we express all co-ordinates in units of c then we get the *universal* law coinciding with the potential of PSL with length equal to 2.

III. RECTANGLE

From Eq.(7) we may get the 3D-potential, at arbitrary point (X, Y, Z) , of a rectangular with constant surface density ρ , with sides of lengths $2c$ and $2b$ along the Z and X axes respectively, with center at origin of co-ordinates X, Y, Z . Equipotential surfaces are symmetrical relative to the rectangle's plane. The "rectangle" may be considered as RP with one infinitesimal dimension dx .

IV. CUBE

From all RPs with three dimensions of RP being *finite*, the case of cube is of the utmost interest. We consider this case in detail. First, the gravitational potential of homogeneous cube is symmetric relative to all three co-ordinates which may have only even powers. Second, there is a scaling invariance: if we express all co-ordinates in units of cube edge half-length, a , then we get the *universal* law coinciding with the gravitational potential of homogeneous cube with edge length equal to 2.

A. Some particular points

Here are values of gravitational potential of homogeneous cube, with density ρ and with edge length $2a$, at some particular points (note that here all potentials are given in units of $a^2 G \rho$):
at the center:

$$U(0, 0, 0) = 24 \ln \frac{1 + \sqrt{3}}{\sqrt{2}} - 2\pi = 9.52017; \quad (11)$$

at the vertex:

$$U(1, 1, 1) = 12 \ln \frac{\sqrt{3} + 1}{\sqrt{2}} - \pi = \frac{1}{2} U(0, 0, 0); \quad (12)$$

at the center of the face:

$$U(0, 0, 1) = 4 \operatorname{arctanh} \frac{157}{129} \sqrt{\frac{2}{3}} - 4 \operatorname{arctan} \frac{7}{3} \sqrt{\frac{2}{3}}; \quad (13)$$

at the middle of the edge:

$$U(0, 1, 1) = 4 \ln 10 - \operatorname{arctan} \frac{4}{3} - 8 \operatorname{arctan} \frac{1}{3} = 4 \ln 10 - \operatorname{arctan} \frac{44}{117} - \pi. \quad (14)$$

Though the solution we discuss is exact however it is rather complex and is difficult to use, so in the next sections we present some serial expansions.

B. Neighborhood of the center

For the homogeneous cube with density ρ and length of edge $2a$, the gravitational potential near the center is spherically-symmetric and quadratic in co-ordinates (note that all co-ordinates are in this paragraph expressed in units of a). The deviation from spherical symmetry appears in the terms of fourth and higher orders:

$$\begin{aligned} U(X, Y, Z) \equiv U_s = a^2 G \rho & \left[\left\{ 24 \ln \frac{1 + \sqrt{3}}{\sqrt{2}} - 2\pi \right\} - \left\{ \frac{2}{3} \pi (X^2 + Y^2 + Z^2) \right\} + \right. \\ & \left\{ -\frac{4}{9\sqrt{3}} (X^4 + Y^4 + Z^4) + \frac{4}{3\sqrt{3}} (X^2 Y^2 + X^2 Z^2 + Y^2 Z^2) \right\} + \left\{ -\frac{1}{162\sqrt{3}} (X^6 + Y^6 + Z^6) + \right. \\ & \left. \left. \frac{5}{108\sqrt{3}} (X^2 Y^4 + X^2 Z^4 + X^4 Y^2 + X^4 Z^2 + Y^2 Z^4 + Y^4 Z^2) - \frac{5}{9\sqrt{3}} X^2 Y^2 Z^2 \right\} \right]. \end{aligned} \quad (15)$$

Here inner braces separate the terms of zeroth, second, fourth and sixth order respectively. Note that no terms with odd powers may occur in serial expansion of the potential of cube and more generally the potential of RP.

C. Potential at main diagonal of cube

The serial expansion of the gravitational potential at the main diagonals where $\operatorname{abs}(X) = \operatorname{abs}(Y) = \operatorname{abs}(Z) = r\sqrt{3}$, and $r = \sqrt{(X^2 + Y^2 + Z^2)}$ is distance from the center) of the homogeneous cube up to r^{20} is as follows:

$$\begin{aligned} W_{diag}(r) = a^2 G \rho \sum_{i=0}^{i=10} c_{2i} \left(\frac{r^2}{3} \right)^i; & c_0 = 24 \ln \frac{1 + \sqrt{3}}{\sqrt{2}} - 2\pi, c_2 = -2\pi, c_4 = \frac{8}{3\sqrt{3}}, \\ c_6 = -\frac{8}{27\sqrt{3}}, c_8 = \frac{136}{567\sqrt{3}}, c_{10} = -\frac{104}{2187\sqrt{3}}, c_{12} = \frac{54392}{1082565\sqrt{3}}, c_{14} = -\frac{136}{19683\sqrt{3}}, \\ c_{16} = \frac{842168}{55801305\sqrt{3}}, c_{18} = -\frac{8632}{4782969\sqrt{3}}, c_{20} = \frac{11003576}{1363146165\sqrt{3}}. \end{aligned} \quad (16)$$

This series represent the actual potential at the main diagonal very accurately; at the "final" point, at the vertex, at $r = \sqrt{3}$, the difference between series and exact solution (4.76016) is only -0.00612 .

D. Potential at co-ordinate axis

Taking $Y = 0$, $Z = 0$ we get potential at X axis which connects the centers of opposite sides. Even in this case exact expression is rather complex, so we present the serial expansion around the center (at point $X = 0$). Series up to X^{20} is as follows:

$$\begin{aligned} U(X, 0, 0) &= a^2 G \rho \sum_{i=0}^{10} c_{2i} X^{2i}; c_0 = 24 \ln \frac{(1+\sqrt{3})}{\sqrt{2}} - 2\pi, c_2 = -\frac{2\pi}{3}, c_4 = -\frac{4}{9\sqrt{3}}, \\ c_6 &= -\frac{1}{162\sqrt{3}}, c_8 = \frac{17}{1701\sqrt{3}}, c_{10} = \frac{13}{104976\sqrt{3}}, c_{12} = -\frac{4307}{6495390\sqrt{3}}, c_{14} = -\frac{17}{5668704\sqrt{3}}, \\ c_{16} &= \frac{174431}{2678462640\sqrt{3}}, c_{18} = \frac{229}{7346640384\sqrt{3}}, c_{20} = -\frac{346867}{43620677280\sqrt{3}}. \end{aligned} \quad (17)$$

E. Approximations by sphere

There are many ways of comparing the cube, with edge length $2a$ and density ρ , with a homogeneous sphere.

1. First naive approximation is by the "inscribed" sphere of radius $R_1 = a$ and with the same *density* ρ . This gives the next rough estimation for the gravitational energy of cube (here $M_1 = 4\pi/3\rho R_1^3$ is the mass of sphere):

$$W_1 = \frac{3}{5} \frac{G M_1^2}{R_1} = \frac{16}{15} \pi^2 G \rho^2 a^5.$$

Note that gravitational potential energy W of any body is of negative sign but in this paper we loosely write all W 's with positive sign.

2. Second approximation is by the "inscribed" sphere with the *mass* equal to cube's mass (density of this sphere will be larger than ρ , density of cube). Then we have the next, less rough, estimation for the gravitational energy of cube (here $M_2 = 8\rho a^3$ is the mass of sphere):

$$W_2 = \frac{3}{5} \frac{G M_2^2}{a} = \frac{192}{5} G \rho^2 a^5.$$

3. Next approximation is by the sphere whose volume equal to cube's volume. Radius R_3 of this "equivolume" sphere is $R_3 = (6/\pi)^{1/3}a$. Homogeneous sphere with this radius and with the density ρ equal to density of cube gives the next approximation for the gravitational energy of cube (here $M_3 = 8\rho a^3$ is the mass of sphere):

$$W_3 = \frac{3}{5} \frac{G M_3^2}{R_3} = \frac{192}{5} \left(\frac{\pi}{6}\right)^{1/3} G \rho^2 a^5 = 30.95 G \rho^2 a^5. \quad (18)$$

Gravitational potential at the center of such sphere, $2\pi G \rho R_3^2 = 2\pi(6/\pi)^{2/3} G \rho a^2 = 9.672 G \rho a^2$ differs from the exact value of central potential of cube (see Eq. (11)) by less than 1.6%. We deduce from this that W_3 (18) should be rather good approximation to gravitational energy of homogeneous cube.

4. Radius R_4 of sphere with the same density and the same central potential as ones of the cube is $[U(0, 0, 0)/2\pi]^{1/2}a$, with $[U(0, 0, 0)]$ given by (11). This gives another good estimation of gravitational energy of cube:

$$W_4 = \frac{3}{5} \frac{G M_4^2}{R_4} = \frac{16}{15} \pi^2 \left(\frac{U(0, 0, 0)}{2\pi} \right)^{5/2} G \rho^2 a^5 = 29.75 G \rho^2 a^5. \quad (19)$$

To calculate the potential energy of homogeneous cube we used Mathematica to integrate numerically the formula (8) over the volume of the cube and obtained the numerical value

$$W_{cube} = 30.117 G \rho^2 a^5. \quad (20)$$

We note that (18) and (19) give very accurate bounds to "exact" value (20).

V. POTENTIAL AT THE CENTER OF RP

We write down the potential in the center of RP:

$$U_{RP}(0,0,0) = 4 * \{ab \ln \frac{d+c}{d-c} + bc \ln \frac{d+a}{d-a} + cd \ln \frac{d+b}{d-b} - a^2 \arctan \frac{bc}{ad} - b^2 \arctan \frac{ac}{bd} - c^2 \arctan \frac{ab}{cd}\}. \quad (21)$$

Here $d = (a^2 + b^2 + c^2)^{1/2}$ is the main diagonal of RP. Even potential in the center of RP could not be scaled by values of a, b, c . Three particular cases are of the larger interest:

a) *cube* corresponding to case $c = b = a$, see (11);

b) *long thin stick with square cross-section* corresponding to case $a \gg b = c$:

$$U_{stick}(0,0,0) = b^2(-8 \ln(b/a) + 12 - 2\pi + 4 \ln 2); \quad (22)$$

c) *thin square plate* corresponding to the case $a \ll b = c$:

$$U_{plate}(0,0,0) = 16 a b \ln(\sqrt{2} + 1) - 2\pi a^2. \quad (23)$$

These formulae will be used further for comparing RP with ellipsoid.

VI. COMPARISON WITH ELLIPSOID

It is interesting to compare the homogeneous RP and the homogeneous triaxial ellipsoid with semi-axes A, B, C , and with density and central value of gravitational potential as ones of RP.

The potential in the center of homogeneous triaxial ellipsoid with semi-axes A, B, C and density ρ is [6] :

$$U_{ell}(0,0,0) = \pi \rho G A B C \int_0^\infty \frac{ds}{\sqrt{(A^2 + s)(B^2 + s)(C^2 + s)}}. \quad (24)$$

The potential energy of triaxial ellipsoid is [6]:

$$W_{ell} = \frac{3}{10} G M^2 \int_0^\infty \frac{ds}{\sqrt{(A^2 + s)(B^2 + s)(C^2 + s)}}. \quad (25)$$

Here $M = 4/3 \pi \rho A B C$ is the ellipsoid's mass.

From (24) and (25), we note the interesting relation between the gravitational potential at the center of homogeneous ellipsoid and the gravitational potential energy of the ellipsoid:

$$W_{ell} = \frac{2}{5} U_{ell}(0,0,0) M_{ell} \quad (26)$$

for any semi-axes!

For the cube we have from (11) and (20) the relation

$$W_{cube} = 0.39544 U_{cube}(0,0,0) M_{cube}, \quad (27)$$

which is very close to (26).

As integral in (24) and (25) in general case is expressed only in terms of elliptic integrals, to simplify calculations we assume $A > B = C$ and also $a > b = c$, then we have for the potential energy of ellipsoid of revolution [6]:

$$W_{ell} = \frac{3}{5} \frac{GM^2}{\sqrt{(A^2 - B^2)}} \operatorname{arccosh} \frac{A}{B}. \quad (28)$$

and for potential at the center:

$$U_{ell}(0,0,0) = \frac{5}{2 M_{ell}} W_{ell}. \quad (29)$$

Here $M = 4/3 \pi \rho A B^2$ is the mass of the ellipsoid of revolution.

There is no simple relation between $U(0,0,0)$ and W for RP with arbitrary values of edge lengths and we may use relation (26) for rather precise estimation of potential energy of RP with known $U(0,0,0)$ from (21).

Potential at vertex of RP

Potential of RP at vertex is: $U(A, B, C) = 1/2 U(0, 0, 0)$ that is exactly half of the potential at center of RP!

VII. POTENTIAL ENERGY OF RP

To get the potential energy of RP, WRP , we need to calculate triple integral over the volume of body:

$$WRP = \frac{1}{2} \rho \int_{-a}^a dX \int_{-b}^b dY \int_{-c}^c U(X, Y, Z, a, b, c) dZ. \quad (30)$$

After some lengthy interactive session with Mathematica, we get the potential energy of homogeneous RP with density ρ and edge lengths $2a$, $2b$, $2c$, which we write down in the following concise form:

$$\begin{aligned} WRP &= G \rho^2 [f(a, b, c) + f(b, c, a) + f(c, a, b)]; f(a, b, c) = c_5 a^5 + c_4 a^4 + c_3 a^3 + c_2 a^2; \\ c_5 &= \frac{32}{15}; c_4 = \frac{32}{15}(d - d1 - d3) - \frac{16b}{3} \ln \frac{(d1-b)(d+b)}{a d3} - \frac{16c}{3} \ln \frac{(d3-c)(d+c)}{a d1}; \\ c_3 &= -\frac{64bc}{3} \arctan \frac{bc}{ad}; d = \sqrt{a^2 + b^2 + c^2}; d1 = \sqrt{a^2 + b^2}; d3 = \sqrt{a^2 + c^2}; \\ c_2 &= \frac{32b^2}{5}(d1 - d) + \frac{32c^2}{5}(d3 - d) - 16bc^2 \ln \frac{d-b}{d+b} - 16b^2c \ln \frac{d-c}{d+c}. \end{aligned} \quad (31)$$

A. Potential energy of cube

From (31), taking $c = a$, $b = a$, we get the potential energy of homogeneous cube with edge length $2a$:

$$WC = 32G\rho^2 a^5 \left\{ \frac{2\sqrt{3} - \sqrt{2} - 1}{5} + \frac{\pi}{3} + \ln[(\sqrt{2} - 1)(2 - \sqrt{3})] \right\} = 30.117G\rho^2 a^5. \quad (32)$$

B. Potential energy of thin long stick

We consider a case $a \gg b = c$ that is a case of thin long stick with square cross-section. Leading term in expansion of WRP (31) gives the potential energy of thin long stick:

$$W_{stick} = \frac{32}{3} G \rho^2 a b^4 \ln \frac{a}{b}. \quad (33)$$

From this and (22) we get for stick:

$$\frac{W_{stick}}{8ab^2 U_{stick}(0, 0, 0)} = \frac{1}{2}. \quad (34)$$

C. Potential energy of thin square plate

Taking one of RP dimension infinitesimally small, $a \rightarrow 0$, we get, from Eq.(31), the potential energy of thin rectangular plate. Note that first non-zero term in expansion is *quadratic* in a . If we additionally take $b = c$, then we get the potential energy of thin square plate:

$$WS = 64G\rho^2 b^3 a^2 \left(\ln(\sqrt{2} + 1) - \frac{\sqrt{2} - 1}{3} \right) = 47.5714 G \rho^2 b^3 a^2. \quad (35)$$

From (23) and (35) we have another limit for relation WUM:

$$\frac{WS}{8ab^2 U_{pl}(0, 0, 0)} = \frac{1}{2} - \frac{\sqrt{2} - 1}{6 \ln(\sqrt{2} + 1)} = .421673. \quad (36)$$

D. Relation between potential energy, gravitational potential and mass of RP

General picture of relation between potential energy, gravitational potential and mass of RP is shown in the Fig.1. We shortly denote it as WUM which means: "potential energy W /(potential at the center $U(0)$ x mass of RP M)". For homogeneous ellipsoid $WUM = 2/5$, see (26).

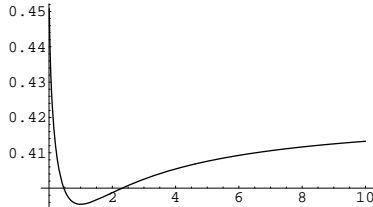


FIG. 1. WUM for RP with square cross section $2b \times 2b$ and length $2a$. Abscissas are values of b/a and ordinates are values of relation $WUM = W/U(0)M$ that is ratio of gravitational potential energy of RP to product of gravitational potential at the center of RP by mass of homogeneous RP. WUM has minimum for cube ($b/a = 1$), tends to $1/2$ at $b/a \rightarrow 0$ (thin long stick) and to $\frac{1}{2} - \frac{\sqrt{2}-1}{6 \ln(\sqrt{2}+1)} = .421673$ at $b/a \rightarrow \infty$ (thin square plate). For homogeneous ellipsoid $WUM = 2/5$.

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- [1] G. N. Duboshin. Celestial mechanics. Nauka, Moskva, 1964.
G. N. Duboshin. Theory of gravitation. Fizmatgiz, Moskva, 1961.
 - [2] B.P. Kondrat'ev, Astron. Rep. **37**(2), 295, 1993;
B.P. Kondrat'ev and V.A. Antonov, Astron. Rep. **37**(3), 300, 1993.
 - [3] Z.F. Seidov, P.I. Skvirsky (in preparation)
 - [4] B.M. Budak, S.V. Fomin. Multiple integrals, field theory and series. Mir, Moscow, 1973;
B.M. Budak, A.A. Samarskii. A.N. Tikhonov, Sbornik zadach po matematicheskoy fizike. Nauka, Moskva, 1972.
 - [5] S. Wolfram. Mathematica Book. Addison-Wesley PC,N.Y. 1999.
 - [6] L.D. Landau and E.M. Lifshitz. The classical theory of fields. Pergamon Press, Oxford, 1975.
 - [7] M. Pick, J. Picha, V. Vyskochil. Theory of the Earth's gravitational field. Elsevier, NY, 1973.
W.D. Mac-Millan. The theory of potential. Dover Pub. N.Y.1958;
The next useful references on the subject (mainly external potential of homogeneous bodies) have been kindly communicated to authors by M. Trott.
D. Nagy. Geophysics **31**, 362, 1966;
J.Waldrogl. ZAMP **27**,867,, 1976; **30**, 388, 1979;
G. Balmino. Celest. Mech. and Dynam. Astron. **60**, 331, 1994;
R. A. Werner. Celest. Mech. and Dynam. Astron. **59**, 253, 1994; **65**,313.
C.C. Grosjean. SIAM Review **38**, 515, 1996.

APPENDIX

Here we present a full explicit expression for gravitational potential of homogeneous rectangular parallelepiped with density ρ and with lengths of edges a, b, c along X, Y and Z axes respectively; X, Y, Z are Cartesian co-ordinates of point at which the gravitational potential is calculated, and the origin of co-ordinates coincides with the center of parallelepiped. This expression is valid for any point X, Y, Z inside as well as outside the body:

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U(X,Y,Z,a,b,c)=
{(-c-Z)^2*ArcTan[((-a-X)*(-b-Y))/(Sqrt[(-a-X)^2+(-b-Y)^2+(-c-Z)^2]*(-c-Z))]-
(-c-Z)^2*ArcTan[((a-X)*(-b-Y))/(Sqrt[(a-X)^2+(-b-Y)^2+(-c-Z)^2]*(-c-Z))]-
(-c-Z)^2*ArcTan[((-a-X)*(b-Y))/(Sqrt[(-a-X)^2+(b-Y)^2+(-c-Z)^2]*(-c-Z))]+
(-c-Z)^2*ArcTan[((a-X)*(b-Y))/(Sqrt[(a-X)^2+(b-Y)^2+(-c-Z)^2]*(-c-Z))]+
(-b-Y)^2*ArcTan[((-a-X)*(-c-Z))/((-b-Y)*Sqrt[(-a-X)^2+(-b-Y)^2+(-c-Z)^2])]+
(-a-X)^2*ArcTan[((-b-Y)*(-c-Z))/((-a-X)*Sqrt[(-a-X)^2+(-b-Y)^2+(-c-Z)^2])]-
(-b-Y)^2*ArcTan[((a-X)*(-c-Z))/((-b-Y)*Sqrt[(a-X)^2+(-b-Y)^2+(-c-Z)^2])]-
(a-X)^2*ArcTan[((-b-Y)*(-c-Z))/((-a-X)*Sqrt[(a-X)^2+(-b-Y)^2+(-c-Z)^2])]-
(b-Y)^2*ArcTan[((-a-X)*(-c-Z))/((-b-Y)*Sqrt[(-a-X)^2+(b-Y)^2+(-c-Z)^2])]-
(-a-X)^2*ArcTan[((b-Y)*(-c-Z))/((-a-X)*Sqrt[(-a-X)^2+(b-Y)^2+(-c-Z)^2])]+
(b-Y)^2*ArcTan[((a-X)*(-c-Z))/((-b-Y)*Sqrt[(a-X)^2+(b-Y)^2+(-c-Z)^2])]+
(a-X)^2*ArcTan[((b-Y)*(-c-Z))/((a-X)*Sqrt[(a-X)^2+(b-Y)^2+(-c-Z)^2])]-
(c-Z)^2*ArcTan[((-a-X)*(b-Y))/(Sqrt[(-a-X)^2+(-b-Y)^2+(c-Z)^2]*(c-Z))]+
(c-Z)^2*ArcTan[((a-X)*(b-Y))/(Sqrt[(a-X)^2+(-b-Y)^2+(c-Z)^2]*(c-Z))]+
(c-Z)^2*ArcTan[((-a-X)*(c-Z))/((-b-Y)*Sqrt[(-a-X)^2+(b-Y)^2+(c-Z)^2]*(c-Z))]-
(c-Z)^2*ArcTan[((a-X)*(c-Z))/((-b-Y)*Sqrt[(a-X)^2+(b-Y)^2+(c-Z)^2]*(c-Z))]-
(-b-Y)^2*ArcTan[((-a-X)*(c-Z))/((-b-Y)*Sqrt[(-a-X)^2+(-b-Y)^2+(c-Z)^2])]-
(-a-X)^2*ArcTan[((-b-Y)*(c-Z))/((-a-X)*Sqrt[(-a-X)^2+(-b-Y)^2+(c-Z)^2])]+
(-b-Y)^2*ArcTan[((a-X)*(c-Z))/((-b-Y)*Sqrt[(a-X)^2+(-b-Y)^2+(c-Z)^2])]+
(a-X)^2*ArcTan[((-b-Y)*(c-Z))/((a-X)*Sqrt[(a-X)^2+(-b-Y)^2+(c-Z)^2])]+
(b-Y)^2*ArcTan[((-a-X)*(c-Z))/((-b-Y)*Sqrt[(-a-X)^2+(b-Y)^2+(c-Z)^2])]+
(-a-X)^2*ArcTan[((b-Y)*(c-Z))/((-a-X)*Sqrt[(-a-X)^2+(b-Y)^2+(c-Z)^2])]-
(b-Y)^2*ArcTan[((a-X)*(c-Z))/((-b-Y)*Sqrt[(a-X)^2+(b-Y)^2+(c-Z)^2])]-
(a-X)^2*ArcTan[((b-Y)*(c-Z))/((a-X)*Sqrt[(a-X)^2+(b-Y)^2+(c-Z)^2])}] / 2 -
(-b-Y)*(-c-Z)*Log[-a-X+Sqrt[(-a-X)^2+(-b-Y)^2+(-c-Z)^2]] -
(-a-X)*(-c-Z)*Log[-b-Y+Sqrt[(-a-X)^2+(-b-Y)^2+(-c-Z)^2]] +
(-b-Y)*(-c-Z)*Log[a-X+Sqrt[(a-X)^2+(-b-Y)^2+(-c-Z)^2]] +
(a-X)*(-c-Z)*Log[-b-Y+Sqrt[(a-X)^2+(-b-Y)^2+(-c-Z)^2]] +
(b-Y)*(-c-Z)*Log[-a-X+Sqrt[(-a-X)^2+(b-Y)^2+(-c-Z)^2]] +
(-a-X)*(-c-Z)*Log[b-Y+Sqrt[(-a-X)^2+(b-Y)^2+(-c-Z)^2]] -
(b-Y)*(-c-Z)*Log[a-X+Sqrt[(a-X)^2+(b-Y)^2+(-c-Z)^2]] -
(a-X)*(-c-Z)*Log[b-Y+Sqrt[(a-X)^2+(b-Y)^2+(-c-Z)^2]] +
(-b-Y)*(c-Z)*Log[-a-X+Sqrt[(-a-X)^2+(-b-Y)^2+(c-Z)^2]] +
(-a-X)*(c-Z)*Log[-b-Y+Sqrt[(-a-X)^2+(-b-Y)^2+(c-Z)^2]] -
(-b-Y)*(c-Z)*Log[a-X+Sqrt[(a-X)^2+(-b-Y)^2+(c-Z)^2]] -
(a-X)*(c-Z)*Log[-b-Y+Sqrt[(a-X)^2+(-b-Y)^2+(c-Z)^2]] -
(b-Y)*(c-Z)*Log[-a-X+Sqrt[(-a-X)^2+(b-Y)^2+(c-Z)^2]] -
(-a-X)*(c-Z)*Log[b-Y+Sqrt[(-a-X)^2+(b-Y)^2+(c-Z)^2]] +
(b-Y)*(c-Z)*Log[a-X+Sqrt[(a-X)^2+(b-Y)^2+(c-Z)^2]] +
(a-X)*(c-Z)*Log[b-Y+Sqrt[(a-X)^2+(b-Y)^2+(c-Z)^2]] -
(-a-X)*(-b-Y)*Log[-c+Sqrt[(-a-X)^2+(-b-Y)^2+(-c-Z)^2]-Z] +
(a-X)*(-b-Y)*Log[-c+Sqrt[(a-X)^2+(-b-Y)^2+(-c-Z)^2]-Z] +
(-a-X)*(b-Y)*Log[-c+Sqrt[(-a-X)^2+(b-Y)^2+(-c-Z)^2]-Z] -
(a-X)*(b-Y)*Log[-c+Sqrt[(a-X)^2+(b-Y)^2+(-c-Z)^2]-Z] +
(-a-X)*(-b-Y)*Log[c+Sqrt[(-a-X)^2+(-b-Y)^2+(c-Z)^2]-Z] -
(a-X)*(-b-Y)*Log[c+Sqrt[(a-X)^2+(-b-Y)^2+(c-Z)^2]-Z] -
(-a-X)*(b-Y)*Log[c+Sqrt[(-a-X)^2+(b-Y)^2+(c-Z)^2]-Z] +
(a-X)*(b-Y)*Log[c+Sqrt[(a-X)^2+(b-Y)^2+(c-Z)^2]-Z] .

```

Here the potential is given in units of $G\rho$ and the Mathematica's language is used except of first "additional" line, $U(X,Y,Z,a,b,c) =$; Mathematica's designations are corresponding to "usual" mathematical formulas as follows: $\text{Log}[x] \equiv \ln(x)$, $\text{Arctan}[x] \equiv \arctan(x)$, $\text{Sqrt}[x] \equiv x^{1/2}$.