Radon transform

$$[1]\quad p(r,\theta)=\int_{-\infty}^{+\infty} f(x,y)\,ds \quad\text{with}\quad \begin{pmatrix} r \\ s \end{pmatrix}=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Fourier and inverse Fourier transform

$$[2]\quad F(u,v)=\int\int_{-\infty}^{+\infty} f(x,y)\cdot\exp\{-2\cdot\pi\cdot i\cdot(u\cdot x+v\cdot y)\}\,dx\,dy$$

$$\begin{aligned}[3]\quad F(\rho\cdot\cos\theta,\ \rho\cdot\sin\theta) &= \int\int_{-\infty}^{+\infty} f(x,y)\cdot\exp\{-2\cdot\pi\cdot i\cdot\rho\cdot(x\cdot\cos\theta+y\cdot\sin\theta)\}\,dx\,dy \\ &= \int\int_{-\infty}^{+\infty} f(x,y)\cdot\exp(-2\cdot\pi\cdot i\cdot\rho\cdot r)\,dr\,ds \\ &= \int_{-\infty}^{+\infty} p(r,\theta)\cdot\exp(-2\cdot\pi\cdot i\cdot\rho\cdot r)\,dr \quad\leftarrow\text{"Fourier slice theorem"}\end{aligned}$$

$$\begin{aligned}[4]\quad p(r,\theta) &= \int_{-\infty}^{+\infty} F(\rho\cdot\cos\theta,\ \rho\cdot\sin\theta)\cdot\exp(2\cdot\pi\cdot i\cdot r\cdot\rho)\,d\rho \\ &= \int\int\int_{-\infty}^{+\infty} f(x,y)\cdot\exp\{2\cdot\pi\cdot i\cdot(r-x\cdot\cos\theta-y\cdot\sin\theta)\cdot\rho\}\,dx\,dy\,d\rho\end{aligned}$$

Nyquist frequency $\equiv \dfrac{1}{2\cdot\Delta_r}$

$$\begin{aligned}[5]\quad & \int_{-\infty}^{+\infty} \exp\{2\cdot\pi\cdot i\cdot(r-x\cdot\cos\theta-y\cdot\sin\theta)\cdot\rho\}\,d\rho \\ &\approx \int_{-\frac{1}{2\cdot\Delta_r}}^{+\frac{1}{2\cdot\Delta_r}} \exp\{2\cdot\pi\cdot i\cdot(r-x\cdot\cos\theta-y\cdot\sin\theta)\cdot\rho\}\,d\rho \\ &= \left[\frac{\exp\{2\cdot\pi\cdot i\cdot(r-x\cdot\cos\theta-y\cdot\sin\theta)\cdot\rho\}}{2\cdot\pi\cdot i\cdot(r-x\cdot\cos\theta-y\cdot\sin\theta)}\right]_{\rho=-\frac{1}{2\cdot\Delta_r}}^{\rho=+\frac{1}{2\cdot\Delta_r}} \\ &= \frac{2\cdot i\cdot\sin\left(2\cdot\pi\cdot(r-x\cdot\cos\theta-y\cdot\sin\theta)\cdot\frac{1}{2\cdot\Delta_r}\right)}{2\cdot\pi\cdot i\cdot(r-x\cdot\cos\theta-y\cdot\sin\theta)} \\ &= \frac{1}{\Delta_r}\cdot\frac{\sin\{\pi\cdot(r-x\cdot\cos\theta-y\cdot\sin\theta)/\Delta_r\}}{\pi\cdot(r-x\cdot\cos\theta-y\cdot\sin\theta)/\Delta_r} \\ &= \frac{1}{\Delta_r}\cdot\operatorname{sinc}\left(\pi\cdot\frac{r-x\cdot\cos\theta-y\cdot\sin\theta}{\Delta_r}\right)\end{aligned}$$

$$[6]\quad p(r,\theta)\approx\frac{1}{\Delta_r}\cdot\int\int_{-\infty}^{+\infty} f(x,y)\cdot\operatorname{sinc}\left(\pi\cdot\frac{r-x\cdot\cos\theta-y\cdot\sin\theta}{\Delta_r}\right)dx\,dy$$

numerical integration

$$[7]\quad f_{j,k}\equiv f(\Delta_{xy}\cdot X_j,\ \Delta_{xy}\cdot Y_k)\quad\text{where}\quad\left\{\begin{array}{l} X_j\equiv j-C_x \\ Y_k\equiv k-C_y \end{array}\right\}\quad\text{for}\quad\left\{\begin{array}{l} j=0,\cdots,N_x-1 \\ k=0,\cdots,N_y-1 \end{array}\right.$$

$$[8]\quad p(r,\theta)\approx\frac{\Delta_{xy}^2}{\Delta_r}\cdot\sum_{j=0}^{N_x-1}\sum_{k=0}^{N_y-1} f_{j,k}\cdot\operatorname{sinc}\left(\pi\cdot\frac{r-\Delta_{xy}\cdot(X_j\cdot\cos\theta+Y_k\cdot\sin\theta)}{\Delta_r}\right)$$

when $\Delta_r=\Delta_{xy}$ and $R_l\equiv l-C_r$ for $l=0,\cdots,N_r-1$

$$[9]\quad p_l(\theta)\equiv p(\Delta_r\cdot R_l,\ \theta)\approx\Delta_{xy}\cdot\sum_{j=0}^{N_x-1}\sum_{k=0}^{N_y-1} f_{j,k}\cdot\operatorname{sinc}\{\pi\cdot(R_l-X_j\cdot\cos\theta-Y_k\cdot\sin\theta)\}$$

[1] $$p(a,b)=\int_{-\infty}^{+\infty} f(x,y,z)\,dc \quad \text{with} \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} R_{ax} & R_{ay} & R_{az} \\ R_{bx} & R_{by} & R_{bz} \\ R_{cx} & R_{cy} & R_{cz} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \leftarrow \text{simple coordinate rotation}$$

[2] $$F(u,v,w)=\int\int\int_{-\infty}^{+\infty} f(x,y,z)\cdot \exp\{-2\cdot\pi\cdot i\cdot(u\cdot x+v\cdot y+w\cdot z)\}\,dx\,dy\,dz$$

[3] $$\begin{aligned} &F(\alpha\cdot R_{ax}+\beta\cdot R_{bx},\ \alpha\cdot R_{ay}+\beta\cdot R_{by},\ \alpha\cdot R_{az}+\beta\cdot R_{bz}) \\ &= \int\int\int_{-\infty}^{+\infty} f(x,y,z)\cdot\exp\{-2\cdot\pi\cdot i\cdot(\alpha\cdot(R_{ax}\cdot x+R_{ay}\cdot y+R_{az}\cdot z) \\ &\qquad\qquad +\beta\cdot(R_{bx}\cdot x+R_{by}\cdot y+R_{bz}\cdot z))\}\,dx\,dy\,dz \\ &= \int\int\int_{-\infty}^{+\infty} f(x,y,z)\cdot\exp\{-2\cdot\pi\cdot i\cdot(\alpha\cdot a+\beta\cdot b)\}\,da\,db\,dc \\ &= \int\int_{-\infty}^{+\infty} p(a,b)\cdot\exp\{-2\cdot\pi\cdot i\cdot(\alpha\cdot a+\beta\cdot b)\}\,da\,db \end{aligned}$$

[4] $$\begin{aligned} p(s,t) &= \int\int_{-\infty}^{+\infty} F(\alpha\cdot R_{ax}+\beta\cdot R_{bx},\ \alpha\cdot R_{ay}+\beta\cdot R_{by},\ \alpha\cdot R_{az}+\beta\cdot R_{bz})\cdot\exp\{2\cdot\pi\cdot i\cdot(s\cdot\alpha+t\cdot\beta)\}\,d\alpha\,d\beta \\ &= \int\int\int\int\int_{-\infty}^{+\infty} f(x,y,z)\cdot\exp\{2\cdot\pi\cdot i\cdot((s-a)\cdot\alpha+(t-b)\cdot\beta)\}\,dx\,dy\,dz\,d\alpha\,d\beta \\ &= \int\int\int_{-\infty}^{+\infty} f(x,y,z)\cdot\int_{-\infty}^{+\infty}\exp\{2\cdot\pi\cdot i\cdot(s-a)\cdot\alpha\}\,d\alpha \\ &\qquad \cdot\int_{-\infty}^{+\infty}\exp\{2\cdot\pi\cdot i\cdot(t-b)\cdot\beta\}\,d\beta\,dx\,dy\,dz \end{aligned}$$

[5] $$\int_{-\infty}^{+\infty}\exp\{2\cdot\pi\cdot i\cdot(s-a)\cdot\alpha\}\,d\alpha \approx \int_{-\frac{1}{2\cdot\Delta_a}}^{+\frac{1}{2\cdot\Delta_a}}\exp\{2\cdot\pi\cdot i\cdot(s-a)\cdot\alpha\}\,d\alpha = \frac{1}{\Delta_a}\cdot\text{sinc}\left(\pi\cdot\frac{s-a}{\Delta_a}\right)$$

$$\int_{-\infty}^{+\infty}\exp\{2\cdot\pi\cdot i\cdot(t-b)\cdot\beta\}\,d\beta \approx \int_{-\frac{1}{2\cdot\Delta_b}}^{+\frac{1}{2\cdot\Delta_b}}\exp\{2\cdot\pi\cdot i\cdot(t-b)\cdot\beta\}\,d\beta = \frac{1}{\Delta_b}\cdot\text{sinc}\left(\pi\cdot\frac{t-b}{\Delta_b}\right)$$

[6] $$p(s,t)\approx\frac{1}{\Delta_a\cdot\Delta_b}\cdot\int\int\int_{-\infty}^{+\infty} f(x,y,z)\cdot\text{sinc}\left(\pi\cdot\frac{s-a}{\Delta_a}\right)\cdot\text{sinc}\left(\pi\cdot\frac{t-b}{\Delta_b}\right)dx\,dy\,dz$$

[7] $$f_{j,k,l}\equiv f(\Delta_a\cdot X_j,\ \Delta_a\cdot Y_k,\ \Delta_b\cdot Z_l) \quad \text{where} \quad \left\{\begin{array}{c} X_j\equiv j-C_x \\ Y_k\equiv k-C_y \\ Z_l\equiv l-C_z \end{array}\right\} \quad \text{for} \quad \left\{\begin{array}{l} j=0,\cdots,N_x-1 \\ k=0,\cdots,N_y-1 \\ l=0,\cdots,N_z-1 \end{array}\right.$$

[8] $$p(s,t)\approx\Delta_a\cdot\sum_{j,k,l} f_{j,k,l}\cdot\text{sinc}\left(\pi\cdot\frac{s-a_{j,k,l}}{\Delta_a}\right)\cdot\text{sinc}\left(\pi\cdot\frac{t-b_{j,k,l}}{\Delta_b}\right)$$

$$\text{where} \quad \left\{\begin{array}{l} a_{j,k,l}\equiv R_{ax}\cdot\Delta_a\cdot X_j+R_{ay}\cdot\Delta_a\cdot Y_k+R_{az}\cdot\Delta_b\cdot Z_l \\ b_{j,k,l}\equiv R_{bx}\cdot\Delta_a\cdot X_j+R_{by}\cdot\Delta_a\cdot Y_k+R_{bz}\cdot\Delta_b\cdot Z_l \end{array}\right.$$

[9] $$\begin{aligned} p_{m,n}\equiv p(\Delta_a\cdot A_m,\ \Delta_b\cdot B_n)\approx\Delta_a\cdot\sum_{j,k,l} f_{j,k,l}\cdot\text{sinc}\left\{\pi\cdot\left(A_m-(R_{ax}\cdot X_j+R_{ay}\cdot Y_k)-\frac{\Delta_b}{\Delta_a}\cdot R_{az}\cdot Z_l\right)\right\} \\ \cdot\,\text{sinc}\left\{\pi\cdot\left(B_n-\frac{\Delta_a}{\Delta_b}\cdot(R_{bx}\cdot X_j+R_{by}\cdot Y_k)-R_{bz}\cdot Z_l\right)\right\} \end{aligned}$$

$$\text{where} \quad \left\{\begin{array}{c} A_m\equiv m-C_a \\ B_n\equiv n-C_b \end{array}\right\} \quad \text{for} \quad \left\{\begin{array}{l} m=0,\cdots,N_a-1 \\ n=0,\cdots,N_b-1 \end{array}\right.$$

Fourier transform of delta function , $\delta(r)$

$$\int_{-\infty}^{+\infty} \delta(r) \cdot \exp(-2 \cdot \pi \cdot i \cdot \rho \cdot r)\, dr = 1$$

$$\rightarrow \quad \delta(r) = \int_{-\infty}^{+\infty} \exp(2 \cdot \pi \cdot i \cdot r \cdot \rho)\, d\rho \quad \leftarrow \text{ inverse Fourier transform}$$

$$\approx \int_{-\frac{1}{2 \cdot \Delta_r}}^{+\frac{1}{2 \cdot \Delta_r}} \exp(2 \cdot \pi \cdot i \cdot r \cdot \rho)\, d\rho = \frac{1}{\Delta_r} \cdot \text{sinc}\left(\pi \cdot \frac{r}{\Delta_r}\right) \qquad [5]$$

2D Radon transform

$$[1] \quad p(r, \theta) = \int_{-\infty}^{+\infty} f(x, y)\, ds \quad \text{with} \quad \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\rightarrow \quad p(r, \theta) = \int \int_{-\infty}^{+\infty} f(x, y) \cdot \delta(r - x \cdot \cos\theta - y \cdot \sin\theta)\, dx\, dy \qquad [1]$$

$$= \int \int_{-\infty}^{+\infty} f(x, y) \cdot \int_{-\infty}^{+\infty} \exp\{2 \cdot \pi \cdot i \cdot (r - x \cdot \cos\theta - y \cdot \sin\theta) \cdot \rho\}\, d\rho\, dx\, dy \qquad [4]$$

$$\approx \frac{1}{\Delta_r} \cdot \int \int_{-\infty}^{+\infty} f(x, y) \cdot \text{sinc}\left(\pi \cdot \frac{r - x \cdot \cos\theta - y \cdot \sin\theta}{\Delta_r}\right) dx\, dy \qquad [6]$$

$$\approx \frac{\Delta_{xy}^2}{\Delta_r} \cdot \sum_{j=0}^{N_x - 1} \sum_{k=0}^{N_y - 1} f_{j,k} \cdot \text{sinc}\left(\pi \cdot \frac{r - \Delta_{xy} \cdot (X_j \cdot \cos\theta + Y_k \cdot \sin\theta)}{\Delta_r}\right) \qquad [8]$$

$$[3] \quad F(\rho \cdot \cos\theta,\ \rho \cdot \sin\theta) = \int_{-\infty}^{+\infty} p(r, \theta) \cdot \exp(-2 \cdot \pi \cdot i \cdot \rho \cdot r)\, dr \quad \leftarrow \text{ "Fourier slice theorem"} \qquad [-]$$

3D Radon transform

$$[1] \quad p(a, b) = \int_{-\infty}^{+\infty} f(x, y, z)\, dc \quad \text{with} \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} R_{ax} & R_{ay} & R_{az} \\ R_{bx} & R_{by} & R_{bz} \\ R_{cx} & R_{cy} & R_{cz} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \leftarrow \text{ simple coordinate rotation}$$

$$\rightarrow \quad p(s, t) = \int \int \int_{-\infty}^{+\infty} f(x, y, z) \cdot \delta(s - R_{ax} \cdot x - R_{ay} \cdot y - R_{az} \cdot z) \cdot \delta(t - R_{bx} \cdot x - R_{by} \cdot y - R_{bz} \cdot z)\, dx\, dy\, dz \qquad [1]$$

$$= \int \int \int_{-\infty}^{+\infty} f(x, y, z) \cdot \int_{-\infty}^{+\infty} \exp\{2 \cdot \pi \cdot i \cdot (s - a) \cdot \alpha\}\, d\alpha \cdot \int_{-\infty}^{+\infty} \exp\{2 \cdot \pi \cdot i \cdot (t - b) \cdot \beta\}\, d\beta\, dx\, dy\, dz \qquad [4]$$

$$\approx \frac{1}{\Delta_a \cdot \Delta_b} \cdot \int \int \int_{-\infty}^{+\infty} f(x, y, z) \cdot \text{sinc}\left(\pi \cdot \frac{s - a}{\Delta_a}\right) \cdot \text{sinc}\left(\pi \cdot \frac{t - b}{\Delta_b}\right) dx\, dy\, dz \qquad [6]$$

$$\approx \Delta_a \cdot \sum_{j,k,l} f_{j,k,l} \cdot \text{sinc}\left(\pi \cdot \frac{s - a_{j,k,l}}{\Delta_a}\right) \cdot \text{sinc}\left(\pi \cdot \frac{t - b_{j,k,l}}{\Delta_b}\right) \qquad [8]$$

$$\text{where} \quad \begin{cases} a_{j,k,l} \equiv R_{ax} \cdot \Delta_a \cdot X_j + R_{ay} \cdot \Delta_a \cdot Y_k + R_{az} \cdot \Delta_b \cdot Z_l \\ b_{j,k,l} \equiv R_{bx} \cdot \Delta_a \cdot X_j + R_{by} \cdot \Delta_a \cdot Y_k + R_{bz} \cdot \Delta_b \cdot Z_l \end{cases}$$

$$[3] \quad F(\alpha \cdot R_{ax} + \beta \cdot R_{bx},\ \alpha \cdot R_{ay} + \beta \cdot R_{by},\ \alpha \cdot R_{az} + \beta \cdot R_{bz}) = \int \int_{-\infty}^{+\infty} p(a, b) \cdot \exp\{-2 \cdot \pi \cdot i \cdot (\alpha \cdot a + \beta \cdot b)\}\, da\, db \qquad [-]$$